Department of Finance & Banking

Weak Euler Scheme for Stochastic Differential Equations with Applications in Finance

Bilal Aslam

This thesis is presented for the Degree of
Master of Philosophy (Economics & Finance)
of
Curtin University

May 2019

DECLARATION

To the best of the author's knowledge, this thesis contains no previously published content or material by any other person except where diligent acknowledgment has been made. Also, the thesis does not contain any material which has been accepted for the award of any other certificate, diploma, or degree in any other university.

In addition, the proposed research study did not require any human research ethics approval from the Curtin University Human Research Ethics Committee.

Bilal Aslam
MPhil Candidate
May 1, 2019

ACKNOWLEDGMENT

This research was supported by the Curtin Sarawak Collaborative Research Scheme. My gratitude also goes to the Faculty of Business and the Graduate School of Curtin University, Malaysia for providing a family-like environment for conducting research.

I wish to express my profound appreciation to Dr. Kenneth Ray Szulczyk, Dr. Ching Seng Yap, and Dr. Changyong Zhang (PI of the funded project) for their continuous support and constructive feedback, as well as the thesis chairperson Dr. Chai Lee Goi for all the coordination, in the course of my study. Furthermore, I am also thankful to the two internal examiners, Dr. Rubaiyat Ahsan Bhuiyan and Dr. Oladokun Nafiu Olaniyi, as well as the two external examiners, Dr. Zhengyu Cui (Stevens Institute of Technology) and Dr. Lingfei Li (The Chinese University of Hong Kong), for their insightful feedback and suggestions.

I would also like to thank the supportive and friendly administrative staff and colleagues in the Faculty of Business, Curtin University, Malaysia.

Contents

List of Figures

List of Tables

Abstract

The random behavior of financial quantities is commonly modeled by using stochastic differential equations (SDEs), which are continuous-time stochastic processes. Depending on the complexity of the model, numerical methods are widely adopted to solve the SDEs. In particular, the Euler method is frequently employed to simulate the time series of the underlying asset price, based on discretizing the life of the financial quantity into reasonably small time steps and updating the asset price at each step in line with a financial model. Among other alternatives, Monte Carlo simulation is probably the most suitable one for pricing path-dependent options and the only one in the case of higher dimensions, where a sufficiently large number of sample paths are generated to obtain, for example, the expected payoff of an option.

Among others, options are important and widely-used financial derivatives. In addition to the standard call and put options that are traded on exchanges, a number of exotic options are commonly traded in the over-the-counter market. Among them, Asian options are popular hedging instruments, the most favorites in the commodity market, and appropriate in executive compensation plans due to the reduced price and less exposition to sudden crashes or price hike. They are hence harder to be manipulated than the standard options.

Since the creation of Asian options, substantial studies have been conducted to price them accurately while no significant efforts have been made to improve the benefits that Asian options offer. In this thesis, a new type of path-dependent options, referred to as the average-Asian options, are introduced to further reduce the volatility of the underlying price risk and minimize the option manipulation threat. Euler method is then adopted to discretize the associated SDE, based on which the options are priced by using Monte Carlo simulation method for both the cases when volatility is constant and when it is stochastic during the life of the option. In the latter case, the option prices are calculated by applying the newly proposed Poisson-Diffusion stochastic volatility model.

It is indicated that the average-Asian options are able to reduce the price volatility and the resulting option price is consistently more stable in the practical situations. It is shown by numerical results that the average-Asian option price is uniformly less than the standard option price as well as the Asian option price when the options are granted in-the-money or at-the-money. On average, the average-Asian option is about 49.3% and 5.4% cheaper than the standard and Asian options, respectively, when granted at-the-money. In addition, the option is less sensitive than the corresponding Asian option, both at the front-end and at the back-end price manipulation.

Chapter 1

Introduction

Unlike a spot contract, which is an agreement to buy or sell an asset immediately, a forward (or forward contract) is an agreement to buy or sell an asset at a certain future time for an agreed price. The fundamental difference between forward and futures contracts is that, forwards are traded in the over-the-counter market, whereas, futures are traded on an exchange. The holder of these two derivatives, futures and forward, is obliged to trade the derivatives at the maturity of the contract irrespective of whether the asset price has risen or fallen during the life of the contract. On the other hand, an option grants the holder the right to buy or sell the underlying asset at a specific time in the future for an agreed price. Because of this added feature, a right instead of an obligation, options markets have been tremendously successful. They are popularly used by different types of traders for the purposes of hedging, speculating, and arbitraging.

For companies that require frequent exchange of currencies every year, it would be preferred to avoid the extra risk imposed by the variability in the currency exchange rates. This is where financial derivatives, particularly options are introduced. For such derivatives, sophisticated mathematics is usually applied to price them so that no arbitrage opportunity exists. The first organized option exchange market in the world, Chicago Board Options Exchange (CBOE), started option trading in 1973, the year in which the landmark Black-Scholes option pricing model was published. The model then led to a boom in options trading and provided mathematical legitimacy to the activities of the CBOE and other option markets around the world (MacKenzie, 2008). At present, CBOE is the largest U.S. options exchange with an annual trading volume of around 1.975 billion contracts in 2018 (CBOE, 2019).

In addition to the standard European and American call and put options that are traded on exchanges, there are several exotic options which are normally traded in the over-the-counter (OTC) market. They are created to meet specific needs of a particular business or risk management, which makes them more attractive (Hull, 2015). One typical example of such options is the binary asset-or-nothing call option (Rubinstein & Reiner, 1991), which pays the asset price if the underlying asset price ends up above the strike price and zero otherwise. Exotic options along with other financial derivatives are gaining increasing importance and are traded nowadays in large quantities in the OTC market (Kyprianou, Schoutens, & Wilmott, 2006), which is shown to be much larger than the exchange-traded market (Hull, 2015).

The payoff from the standard European style option depends on the price of the underlying at expiry. There are then chances that the option may be manipulated or may be too expensive. In case that there is a potential to manipulate the price of the underlier, or that some cheaper option is sought, a popular alternative is the Asian option (Boyle & Emanuel, 1980), which is also called the average option and was first successfully priced in 1987 by David Spaughton and Mark Standish when they were in Tokyo, Japan, Asia (Wilmott, 2007). The payoff of an Asian option is determined by the average underlying price during the life of the option. In addition, the payoff structure of an Asian option resembles that of the variable annuity (Bernard, Cui, & Vanduffel, 2017), an insurance contract typically as a long-term investment aimed at generating income for retirement.

1.1 Motivation

1.1.1 Stochastic Differential Equations

Financial quantities, which are the underlying of the options, change with time randomly. The dynamics of these quantities are generally modeled through stochastic differential equations (SDEs). In the case of models driven by drifts and Brownian motions only, analytical solutions may exist. However, after the incorporation of jumps, in particular Lévy processes, analytical solutions could be available for standard European options, while for pricing exotic options, numerical methods are unavoidable (Cont & Tankov, 2004). This results in finding numerical solutions as the only remedy to the problem. Both in academics and practice, most attention has then been focused on discrete time approximation of SDEs.

1.1.2 Numerical Approximations

As a popular numerical approximation method to price options, Monte Carlo simulation is an appropriate method for pricing path dependent financial options (Hull, 2015). In particular, the complexity of the method grows only linearly with respect to number of stochastic variables (Cont & Tankov, 2004). Thus, the method is suitable for stochastic volatility models where in addition to the asset price, volatility is also considered as a stochastic variable. Besides, solving Lévy-driven SDEs using the Monte Carlo simulation method is the only possible option in many applications, since in practice, only little is known about the functionals of the underlying Lévy processes (Papapantoleon, 2008).

As SDEs are continuous-time stochastic processes, the simulations of these SDEs usually involve discrete time steps. The Euler scheme is a method to discretize a continuous-time process into a discrete-time process. The method involves discretizing the life T of a financial quantity into n time steps and updating its value at each step using a financial model based on an SDE. The option values can then be calculated using the Monte Carlo simulation method. For derivatives, portfolio optimization, and other financial applications, weak order of convergence, defined by error in the expected payoff is required (Giles, 2012).

1.1.3 Options and Hedging

The averaging feature in the Asian option reduces the volatility inherent in the option, which makes it less exposed to sudden crashes or rallies in an asset price and is harder to manipulate (Wilmott, 2007). As a result, Asian options are the most popular exotic payout options chosen by U.S. non-financial firms for the purpose of risk management (Bodnar, Hayt, & Marston, 1998). In particular in the commodity market, now a mainstream financial and investment class (Kyriakou, Pouliasis, & Papapostolou, 2016), end users are often exposed to the average prices over time, which makes Asian options of obvious appeal (Wilmott, 2007). These path-dependent Asian options are particularly appropriate to the electricity market, where the contracts are written to supply continuous electricity over the life of the option. It is therefore reasonable for the electricity market to refer to the average price over the period of the contract (Fanelli, Maddalena, & Musti, 2016). Besides, price manipulation by large market participants is harder in the case of an Asian option as compared to a standard option (Chatterjee et al., 2017). This is critically important in the case of thinly traded commodities since it may be possible to manipulate the price on any given day or near option expiry than the average price of the underlying asset (Linetsky, 2004). For firms to mitigate the principal-agent problem, it is highly recommended to suggest that firms should consider granting Asian options instead of standard options as compensation packages (Tian, 2013).

1.2 Research Focus

Financial engineers create exotic options to make them more attractive, for the purpose of risk management, to increase or decrease the sensitivity of uncertain future price risk, or any other particular needs of the business (Hull, 2015). Options are also designed to align the interest of the parties involved, for example, to mitigate the principal-agent problem in a firm (Bernard, Boyle, & Chen, 2016). In this thesis, the focus is on creating a new path-dependant option, calculating the sensitivity of price jumps or manipulations, and pricing using the Monte Carlo simulation method for both the cases when volatility is constant and when it is stochastic during the life of the option.

1.2.1 Research Gap

Since 1987, substantial efforts have been made to price the reliable alternative, i.e., Asian option, to the vanilla counterparts for financial risk management, especially in the market where either the volume is low or the volatility is high (Fanelli, Maddalena, & Musti, 2016). However, except for the power option in executive compensation literature (Bernard, Boyle, & Chen, 2016), not too many notable efforts have been made to look into what made this option so popular: the averaging concept, reduced price, and safeguard from the option manipulation threat. In this thesis as illustrated in Section 4.3, a new path dependent option, referred to as the average-Asian option, is introduced to further reduce the volatility of the asset prices risk and minimize the option manipulation threat.

1.2.2 Research Questions

Based on the above discussion, the questions to be addressed in this thesis are outlined below.

1. How can a new path-dependent average option be developed to improve the benefits offered by Asian option with a reasonable payoff function?

2. To what extent can the proposed average-Asian option be capable of reducing the underlying price volatility and be used to hedge uncertain future price risk more effectively?

3. How can such an option be priced by using Monte Carlo simulation when the Euler method is used as the discretization scheme for SDEs and the volatility is considered as being constant as well as being stochastic?

4. How can the Asian and average-Asian options be priced when jumps are incorporated into the diffusion process?

1.2.3 Research Objectives

The research objectives, in line with the research questions, are then

1. To introduce a new option, referred to as average-Asian option, so as to improve the benefits offered by Asian option.

2. To analyze the expected payoff, prices, and sensitivity of the average-Asian option by using a combination of methods, and to compare them with those of the standard and Asian options.

3. To value the average-Asian option by using Monte Carlo simulation when the discretization scheme for SDEs is the Euler method and the volatility is constant, as well as when the financial model is a coupled SDE and both the asset price and volatility are stochastic.

4. To price the Asian and average-Asian options after the incorporation of jumps into the diffusion process.

1.3 Research Significance

Options are important financial derivatives for risk management that minimizes the adverse effect of asset price jumps and the potential market manipulation threat. Asian options are widely used as hedging instruments for the said purpose. As anything that reduces the up-front premium in an option contract makes it more popular (Wilmott, 2007), the average-Asian option would be of significant practical importance in real life.

In the financial market, the design and analysis of financial derivatives as insurance and risk management products protecting against undesirable and unpredictable conditions is a challenging and demanding area of research (STORE, 2019). Consequently, the design and analysis of the proposed average-Asian option is expected to be a valuable addition to the family of financial derivatives.

1.4 Thesis Structure

Chapter 2 starts with introduction to stochastic process and its applications to Finance, in particular the basic theory of Lévy processes, followed by important Lévy processes including Brownian motion, Poisson process, and compound Poisson process.

In chapter 3, basic stochastic differential equations and their Euler schemes are given. Then popular financial models are explained including the Black-Scholes model, Merton's jump-diffusion model, and stochastic volatility models. Here, considering highly volatile Chicago board options exchange (CBOE) volatility index during the year of 2018, a Poisson-Diffusion model is proposed to represent the dynamics of highly volatile assets where large deviation from the mean price is expected.

In chapter 4, after introducing the standard and Asian options, the popularity of Asian options for the purpose of financial risk management is reenforced. In addition, the usage of Asian option as an executive compensation package is also discussed. Then, a new type of path-dependent options, referred to as average-Asian options, are developed with the purpose of reducing the underlying price volatility and the option prices. It is further demonstrated that the expected value of an average-Asian call option is less than that of the standard and Asian call options in different practical situations. Moreover, the average-Asian option is also less sensitive to price manipulation than the Asian option.

In chapter 5, the three options, standard, Asian, and average-Asian options are first priced by sampling through a tree method as well as the Monte Carlo simulation method, where the volatility is considered constant. Then, these three options are priced by using a stochastic volatility model, where the volatility is given as a stochastic process. Finally, the option prices are calculated for the newly-proposed Poisson-Diffusion stochastic volatility model. The numerical results show that the average-Asian option is less expensive than the standard option in all the cases, while it is also less expensive than the Asian option when granted in-the money or at-the money.

The conclusion, as well as the potential future work, is presented in chapter 6.

Chapter 2

Lévy Processes

A quantity whose value changes through time in an uncertain way is said to follow a stochastic process, a typical example of which is the stock price. For a stochastic process, the mean change per unit time is called the drift rate and the variance per unit time is called the variance rate.

This chapter gives a concise introduction to the class of stochastic processes known as Lévy processes, which are used to model financial quantities such as asset price, interest rate, or currency exchange rate. The fundamental Lévy processes such as Brownian motion, Poisson process, and compound Poisson process are also explained.

2.1 Definition and Properties

2.1.1 Definition

Let $N(t_0)$ be a random value when $t = 0$, and $N(t_1)$ be another random value when $t = 1$, and so on. When $N(t)$ for $t \in [0, \infty)$ is considered collectively, $N(t)$ is said to be a *stochastic process*, or a Random Process.

A stochastic process $L = L_t : t \geq 0$ is called a *Lévy process* if:

1. $L_0 = 0$
2. L has independent and stationary increments
3. L is stochastically continuous i.e., for every $0 \leq t \leq T$ and $\epsilon \geq 0$,

$$\lim_{h \to 0\infty} P(|L_{t+h} - L_t| \geq \epsilon) = 0$$

A Lévy process represents the motion of a quantity whose successive displacements are random as well as independent, and are statistically identical over time intervals of the same size.

Linear drift, a deterministic process, is the simplest Lévy process, and Brownian motion is the only non-deterministic Lévy process with continuous sample paths (Papapantoleon, 2008). Except for Brownian motion, all other Lévy processes have discontinuous paths. Other popular examples are the Poisson and compound Poisson processes.

2.1.2 Properties

There is a strong relationship between Lévy processes and infinitely divisible distributions as the distribution of a Lévy process has the property of infinite divisibility.

The Law P_X of a random variable X is *infinitely divisible*, if for all $n \in N$ there exist independent and identically distributed random variables $X_1^{(1/n)}, X_2^{(1/n)}, ..., X_n^{(1/n)}$ such that (Papapantoleon, 2008)

$$X = X_1^{(1/n)} + X_2^{(1/n)} + ... + X_n^{(1/n)}$$

Equivalently, the law P_X of a random variable X is infinitely divisible if for all $n \in N$ there exists another law $P_{X^{(1/n)}}$ of a random variable $X^{(1/n)}$ such that

$$P_X = P_{X^{(1/n)}} \times P_{X^{(1/n)}} \times .....P_{X^{(1/n)}}$$

Alternatively, an infinitely divisible random variable can be characterized by using its characteristic function.
The law of a random variable X is infinitely divisible, if for all $n \in N$, there exists a random variable X such that

$$\varphi_X(u) = (\varphi_{X^{(1/n)}}(u))^n$$

Furthermore, if $(P_K)_{K \geq 0}$ is a sequence of infinitely divisible laws and $P_K \to P$, then P is also infinitely divisible (Papapantoleon, 2008). Another important result in the concept of infinite divisibility is the following statement.

The law P_X of a random variable X is infinitely divisible if and only if there exists a triplet (b, c, ν), with $b \in \Re$, $c \in \Re_{\geq 0}$ and a measure satisfying $\nu(0) = 0$ and $\int_{\Re}(1 \wedge |x|^2)\nu(dx) < \infty$, such that (Papapantoleon, 2008; Kyprianou, Schoutens, & Wilmott, 2006)

$$E[e^{iuX}] = exp[ibu - \frac{u^2c}{2} + \int_{\Re}(e^{iux} - 1 - iux1_{|x|<1})\nu(dx)]$$

The distribution of a Lévy process is characterized by its characteristic function, which is given by the *Lévy-Khintchine formula* (Papapantoleon, 2008; Kyprianou, Schoutens, & Wilmott, 2006).

Theorem 2.1 (Lévy-Khintchine representation) *If $L = (L_t)_{0 \leq t \leq T}$ is a Lévy process, then its characteristic function $\varphi_X(u)$ is given by*

$$\varphi_X(u) = E[e^{iuL_t}] = e^{t\psi(u)} = exp[t(ibu - \frac{u^2c}{2} + \int_{\Re}(e^{iux} - 1 - iux1_{|x|<1})\nu(dx))]$$

where $\psi(u)$ is the characteristic exponent of L_1, a random variable with an infinitely divisible distribution. Here $b \in \Re$, $c \geq 0$, 1 is the indicator function, and ν is called the Lévy measure of L satisfying the property $\int_{\Re} 1 \wedge |x|^2 \nu(dx) < \infty$.

Consequently, a Lévy process has three independent components: a linear drift, a Brownian motion, and a superposition of independent Poisson processes with different sizes where $\nu(dx)$ represents the rate of arrival of the Poisson process with jump of size x. These three components of a Lévy process and the Lévy-Khintchine representation are thus determined by the Lévy-Khintchine triplet (b, c, ν).

A Lévy process can be decomposed into the sum of independent Lévy processes. This property is known as the *Lévy-Itô decomposition* (Papapantoleon, 2008).

Theorem 2.2 (Lévy-Itô decomposition) *Consider a triplet (b, c, ν) where $b \in \Re$, $c \in \Re_{\geq 0}$ and ν is a measure satisfying $\nu(0) = 0$ and $\int_{\Re} 1 \wedge |x|^2 \nu(dx) < \infty$. Then, there exists a probability space $(\Omega, \mathcal{F}, P)$ on which four independent Lévy processes exist, $L^{(1)}, L^{(2)}, L^{(3)}$ and $L^{(4)}$, where $L^{(1)}$ is a constant drift, $L^{(2)}$ is a Brownian motion, $L^{(3)}$ is a compound Poisson process, and $L^{(4)}$ is a pure jump martingale with an a.s. countable number of jumps of magnitude less than 1 on each finite time step. Taking $L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}$, there exists a probability space on which a Lévy process $L = (L_t)_{0 \leq t \leq T}$ with characteristic exponent*

$$\psi(u) = ibu - \frac{u^2 c}{2} + \int_{\Re} (e^{iux} - 1 - iux 1_{|x|<1}) \nu(dx)$$

for all $u \in \Re$, is defined.

The process defined by $L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}$ is then a Lévy process with triplet (b, c, ν).

2.2 Brownian Motion

The modern Financial Mathematics has roots in the discovery of the Brownian motion (Brown, 1828) in 1827 by Robert Brown, who observed random motion of microscopic particles resulting from their collision with atoms or molecules in a fluid moving with different velocities and in different directions. Louis Bachelier, who developed the theory of option pricing and became pioneer in Financial Mathematics, was the first to introduce Mathematics of Brownian motion in 1900, to compare its trajectories with stock prices behavior, and to calculate option values (Bachelier, 1900). Albert Einstein in 1905 suggested a mathematical model and expressed that the displacement of a Brownian particle is proportional to the square root of the time elapsed (Einstein, 1911). However, Norbert Wiener in 1921 provided the rigorous mathematical construction of standard Brownian motion (Wiener, 1921). Hence, the standard Brownian motion is also called a Wiener process.

The position of a variable following Brownian motion is considered as 0 when the observation starts at time 0. The increment in the variable over disjoint time intervals is continuous and independent. By the central limit theorem of probability theory, the sum of a large number of independent identically distributed random variables is approximately normal, so each increment is assumed to have a normal probability distribution. The mean increment is zero as there is no preferred direction. As the position of a particle spreads out with time, it is assumed that the variance of the increment is proportional to the length of time that the Brownian motion has been observed.

A random process $z(t)$ where $t \in [0, \infty)$ is called a *standard Brownian motion* or Wiener process if:

1. $z(t) = 0$
2. $z(t)$ has independent increments over non-overlapping time. That is, for all $0 \leq t_1 < t_2 < t_3 ... < t_n$ the random variables $z(t_2) - z(t_1), z(t_3) - z(t_2), ..., z(t_n) - z(t_{n-1})$ are independent.
3. The increment over any time interval $t_n - t_{n-1}$ has a normal probability distribution with mean 0 and variance equal to the length of this time interval. That is, for all $0 \leq t_1 < t_2$, $z(t_2) - z(t_1) \sim N(0, t_2 - t_1)$
4. $z(t)$ has continuous sample paths.

The change in a variable following Brownian motion, Δz, during a small period of time Δt is

$$\Delta z = \varepsilon \sqrt{\Delta t} \tag{2.1}$$

where ε has a standardized normal distribution with mean 0 and variance 1. The uncertainty about the value of the variable z at a certain time in the future, as measured by its standard deviation, increases in the scale of the square root of the time duration.

The numerical approximations have been typically based on a time discretization of the life T of the quantity into N time steps of length Δt . The change in the variable z from time 0 to T is hence divided in N small intervals

$$N = T\Delta t$$

The equation 2.1 in discrete terms can then be written as

$$z(T) - z(0) = \sum_{i=1}^{i=N} \varepsilon_i \sqrt{\Delta t},$$

which is used to simulate Brownian motion. In Matlab, normally distributed random numbers are generated through the command `randn`, and the cumulative sum is obtained by using the `cumsum` command.

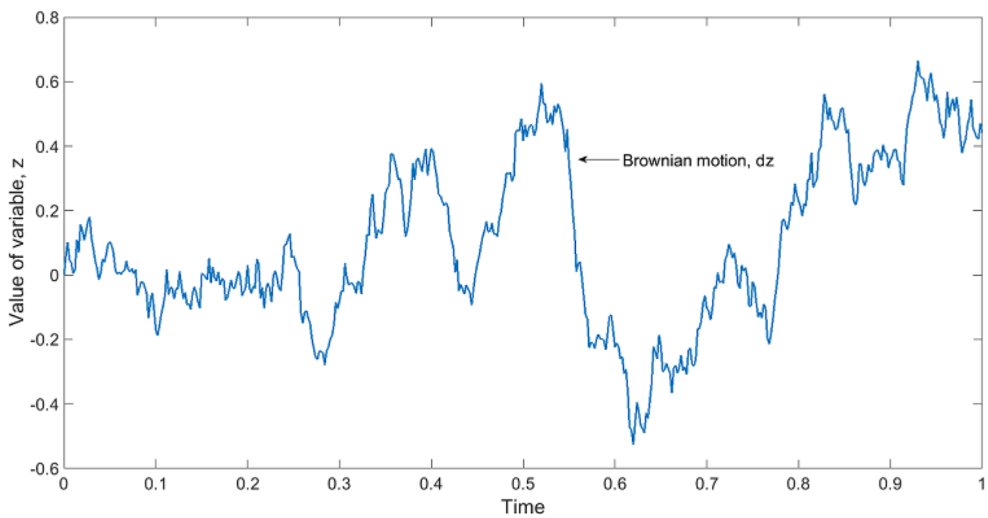

Figure 2.1: A Sample Path of a Brownian Motion

2.2.1 Brownian Motion with Drift

In the limit $\Delta t \to 0$, the change in Brownian motion Δz could be written as dz and a Brownian motion with drift, or generalized Wiener process, for a variable x can be defined in terms of dz as

$$dx = adt + bdz$$

Where a and b are constants.

The Brownian motion with drift has an expected drift rate of a and variance rate of b times a Brownian motion, z. In discrete terms the change Δx in the value of x during a small interval of time Δt is given as

$$\Delta x = a\Delta t + b\varepsilon\sqrt{\Delta t}$$

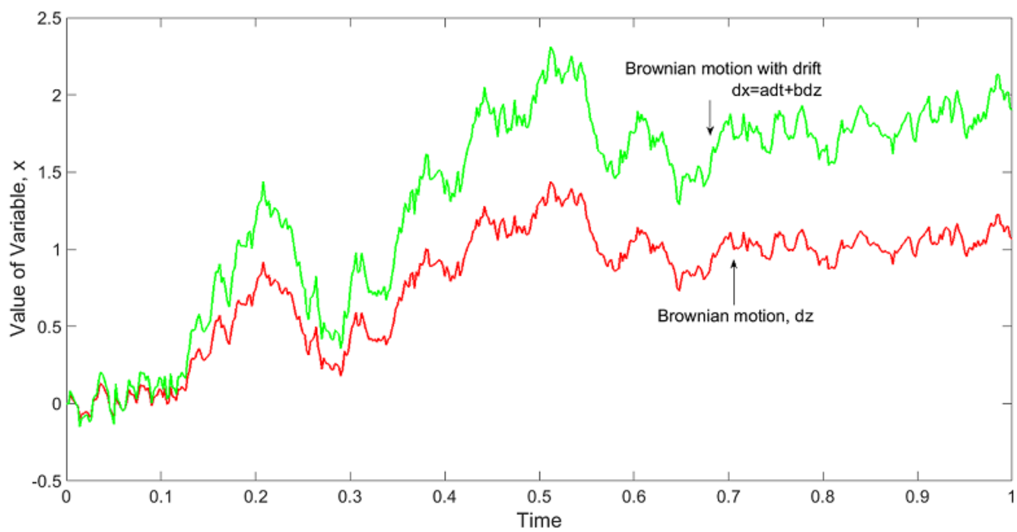

Figure 2.2: Sample Paths of a Brownian Motion dz and a Brownian Motion with Drift dx with $a = 0.3$ and $b = 1.5$

2.2.2 Geometric Brownian Motion

Applying the Brownian motion with drift to stock price S with expected drift rate of μ and a variance rate, or volatility, of σ, gives

$$dS = \mu dt + \sigma dz$$

For stock prices, more precisely, the expected percentage change, rather than the expected absolute change, is constant. Meanwhile, uncertainty, or the volatility, about the future stock prices is proportionate to the current prices. The stock price process then follows

$$dS = \mu S dt + \sigma S dz,$$

which is known as the geometric Wiener process, or geometric Brownian motion (Hull, 2015). It is the most widely used model of stock price behavior. The variable μ is the stock's expected rate of return. The variable σ is the volatility, or standard deviation, of the stock price. The famous Black-Scholes model for option pricing is based on this geometric Brownian motion.

The discrete-time version of the geometric Brownian motion is

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

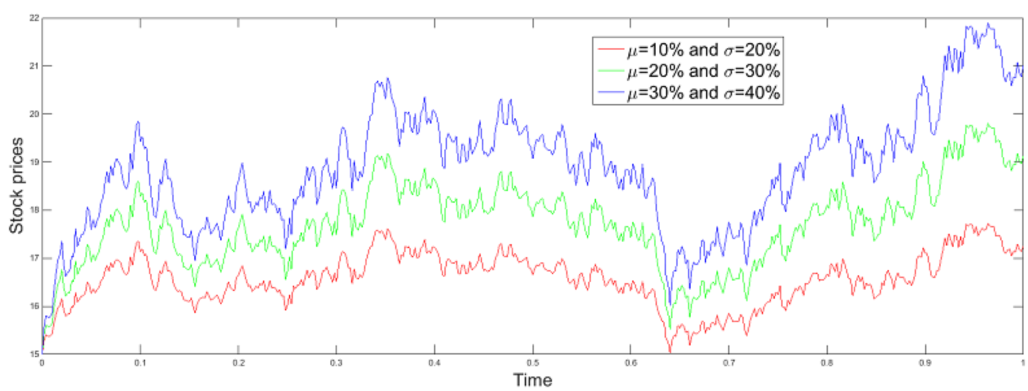

Figure 2.3: Sample Paths of Stock Price Processes with Different μ and σ

2.2.3 Itô Process and Itô's Lemma

An Itô process can be defined as a Brownian motion with drift where the parameters a and b are functions of the underlying variable and time, i.e.,

$$dx = a(x,t)dt + b(x,t)dz$$

Let G be a function of x and t, substituting dx into Itô's lemma (Itō, 1951)

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2 dt$$

gives

$$dG = (\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2)dt + \frac{\partial G}{\partial x}bdz$$

which was used by Black and Scholes in pricing stock options (Black & Scholes, 1973). However, there are some shortfalls in the pricing model. The model assumes that volatility is a known constant, whereas empirical evidences show that the volatility is highly unstable and unpredictable, and so a random variable itself. The model also assumes that an asset price changes continuously and follows a lognormal distribution. However, it has been proven that financial quantities, i.e., stocks, currencies, or interest rates, do not follow the lognormal distribution, but with jumps (Wilmott, 2007; Hull, 2015).

In addition to the aforementioned drawbacks in the Black-Scholes model, Rama Cont studied the statistical issues with asset returns in more detail. The main problem is that log returns on real data exhibit (semi) heavy tails while log returns in the Black-Scholes model are assumed to be normally distributed and hence lightly tailed (Cont, 2001). Among the many suggestions which were proposed to address this particular problem was the simple idea to substitute the use of a Brownian motion with drift by Lévy processes (Kyprianou, Schoutens, & Wilmott, 2006).

2.3 Poisson Process

Sometimes it is important to count number of occurrences or arrivals over time. For example, the number of customers arriving between time t_1 and t_2 in a store, or the number of times a stock price reaching a certain maximum or minimum value. In these cases, a counting process denoted by $N(t)$ is dealt with. The most widely used counting process is Poisson process. It is used in conditions where some occurrences have certain rate (λ) but are completely random. That is, the rate, the average number of times in a certain time period, is known, but not the exact time when it occurs. The occurrences are at random times and follow a Poisson distribution which is closely related to the exponential distribution, which is mainly employed to model the time elapsed between arrivals or events.

A continuous random variable X is assumed to have an exponential distribution with parameter $\lambda > 0$, written as $X \sim \text{Exponential}(\lambda)$, if its probability distribution function (PDF) is given by

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$$

The expected value and variance of the exponential random variable X are given by

$$E[X] = \frac{1}{\lambda} \quad \text{and} \quad Var[X] = \frac{1}{\lambda^2}$$

The cumulative distribution function (CDF) of the exponential random variable X is given by

$$F_X(x) = \int_0^x \lambda e^{-\lambda x} dt = 1 - e^{-\lambda x}, \quad x > 0,$$

which is invertible and its inverse is

$$F_X^{-1} = -\frac{1}{\lambda} \ln(1 - x), \quad \forall x \in [0, 1]$$

A simple consequence is that if U is uniformly distributed on [0,1], then $-\frac{1}{\lambda}\ln U$ is exponentially distributed with parameter λ.
A critical property of the exponential random variable is the characteristic of memorylessness, which states that if X is an exponential random variable with parameter $\lambda > 0$, then X is a memoryless random variable, that is

$$P(X > x + a | X > a) = P(X > x), \quad \text{for} \quad a, x \geq 0$$

The memoryless property says that, it does not matter how long the time has been elapsed. If no arrival has been observed until time a, the distribution of the next arrival from time a is the same as when the observation was started at time zero.

The number of arrivals of an event within a specified time interval has a Poisson distribution with parameter λ if the time elapsed between two successive arrivals of the event has an exponential distribution with parameter λ and it is independent of the previous arrivals. Thus, a Poisson distribution is a discrete probability distribution that exhibits the probability of a given number of arrivals of an event in some time interval if these arrivals occur with a constant rate and independently of the time since the last arrival.

A discrete random variable X is assumed to have a Poisson distribution with parameter $\lambda > 0$, written as $X \sim \text{Poisson}(\lambda)$, if its probability mass function (PMF) is given by

$$P_X(x) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0, 1, 2, 3, ...$$

where $P_X(x)$ is the probability of observing x events in a time interval t. The expected value and variance of the Poisson random variable X are the same, and are given by

$$E[X] = \lambda \quad \text{and} \quad Var[X] = \lambda$$

Specifically, for a fixed rate $\lambda > 0$, a counting process is called a *Poisson process* $N(t), t \in [0, \infty)$ if the following three conditions are held

1. $N(0) = 0$

2. $N(t)$ has independent and stationary occurrences

3. The occurrences in the time interval $t > 0$ has a Poisson(λt) distribution

$$P(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Here λ is the arrival rate, t is the time, and n is the number of the occurrences or arrivals. The arrivals are zero at time $t = 0$, the start of observations, are independent of each other, and follow the Poisson(λt) distribution.

If T is the time until the next arrival, then the probability of the first arrival after time t where $t < T$ is equivalent to that that no event has occurred up to time t. That is,

$$P(T > t) = P(N(t) = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$$

The distribution function of T is thus

$$P(T \leq t) = 1 - e^{-\lambda t}$$

The Poisson process also exhibits memoryless property that means time and events must not be overlapped and must be subtracted in the subsequent counting, i.e.,

$$P(N(t_2) = n_2)|N(t_1) = n_1) = P(N(t_2 - t_1) = n_2 - n_1)$$

Let $N(t)$ be the Poisson process with rate λ and the events be jumps. The first jump occurs at time X_1, second at time X_2, and so on, where X_1, X_2, ... are independent exponential random variables all with the same mean $\frac{1}{\lambda}$, then

$$X_i \sim \text{Exponential}(\lambda), \quad i = 1, 2, 3, ...$$

and the random variables X_i are called interarrival times and are given by

$$S_n = \sum_{i=1}^{n} X_n$$

where S_n is the time of the nth jump, and the Poisson process $N(t)$ counts the number of jumps from time zero to t. In addition, the sum T_n of n independent Exponential(λ) random variables

$$T_n = X_1 + X_2 + ... + X_n$$

is a gamma random variable, i.e.,

$$T_n \sim Gamma(n, \lambda), \quad \text{for } n = 1, 2, 3, ...$$

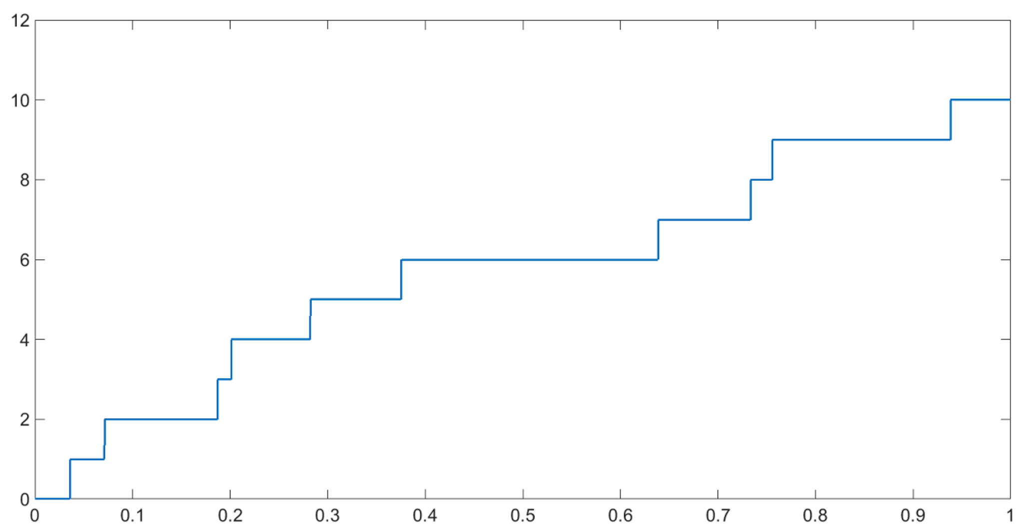

Figure 2.4: A Sample Path of a Poisson Process with $\lambda = 15$

2.3.1 Merging and Splitting Poisson Processes

If $N_1(t), N_2(t), ..., N_m(t)$ are m Poisson processes with rates $\lambda_1, \lambda_2, ..., \lambda_m$ respectively, then the merged $N(t)$, i.e.,

$$N(t) = N_1(t) + N_2(t) + ... + N_m(t)$$

for all $t \in [0, \infty)$ is also a Poisson process with rate $\lambda = \lambda_1 + \lambda_2 + ... + \lambda_m$

Similarly, the Poisson process $N(t)$ could be splitted into many independent Poisson processes. For example, $N_1(t)$ is a Poisson process with rate λp and $N_2(t)$ is a Poisson process with rate $\lambda(1-p)$, where $N_1(t)$ and $N_2(t)$ are independent Poisson processes.

2.3.2 Compound Poisson Process

In a standard Poisson process, jumps are of constant size. Meanwhile, jumps in stock price processes are with random sizes. In compound Poisson process, jumps arrive randomly according to a Poisson process and the size of the jumps is also random with a specified probability distribution. The compound Poisson process with rate $\lambda > 0$ and jump size distribution G is given by

$$Y(t) = \sum_{k=1}^{N(t)} Z_k$$

where $N(t)$ is a Poisson process, and Z_k is a sequence of independent and identically distributed random variables with distribution function G independent of $N(t)$. The jump size

$\Delta Y(t) = Y(t) - Y(t-1)$ at time t is given by the relation

$\Delta Y(t) = Z_{N(t)} \Delta N(t)$

where

$\Delta N(t) = N(t) - N(t-1) \in [0, 1]$

denotes the jump size of the standard Poisson process.

Summary

In this chapter, basic theory of Lévy processes and some fundamental Lévy processes are presented. In addition, the simulations of these processes are also performed. Chapter 3 is devoted to the applications of these processes in financial mathematics.

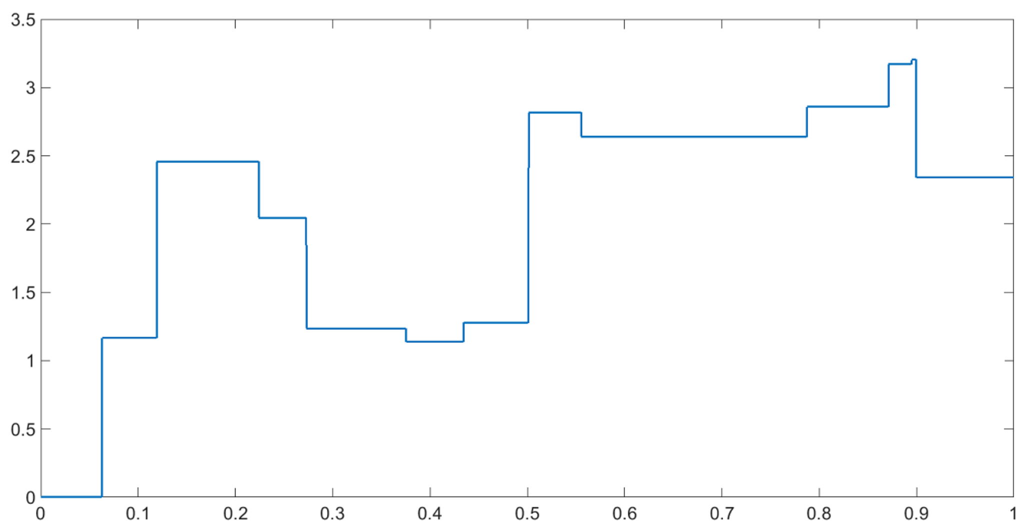

Figure 2.5: A Sample Path of a Compound Poisson Process with $\lambda = 15$ and Normally-Distributed Jump Size

Chapter 3

Stochastic Differential Equations

A variable that changes with time is explained through a differential equation and when a variable changes with time randomly, it is explained through a stochastic differential equation (SDE). The process of writing a stochastic differential equation to describe changes in financial quantities, e.g., asset price, interest rate, currency exchange rate, etc., over time is called financial modelling. Stochastic differential equations have become standard models for financial quantities and their derivatives (Sauer, 2012). An important goal in financial mathematics is to find models for these financial quantities in order to value and hedge derivative securities, value at risk (VaR), and risk management purposes (Kyprianou, Schoutens, & Wilmott, 2006).

This chapter begins with an introduction to the SDE, followed by popular financial models, such as the Black-Scholes, Merton and Kou's jump-diffusion models, and stochastic volatility models. In Section 3.1.6, a Poisson-Diffusion Model to represent the dynamics of highly volatile asset is proposed. Finally in Section 3.2, the Euler scheme for SDEs and some financial models is demonstrated.

Consider a variable X that changes with time randomly and the source of randomness is a Brownian motion. Then, the general form of the SDE is given by

$$dX(t) = \mu\Big[t, X(t)\Big]dt + \sigma\Big[t, X(t)\Big]dz(t)$$

where $dX(t)$ is the stochastic process, and $\mu\Big[t, X(t)\Big]$ is the drift coefficient, the expected drift rate of the variable X. If the drift coefficient is positive, it is called a growth rate and if the drift coefficient is negative, it is called a decay rate. The term $\sigma\Big[t, X(t)\Big]$ is called the diffusion coefficient or volatility. It is the standard deviation or the spread of a distribution. Also, z denotes a Brownian motion. The term $dX(t)$ represents change in the variable X over a very short time interval from t to $t + dt$, that is,

$$dX(t) = X(t + dt) - X(t)$$

Integrating $dX(t)$ from time 0 to T produces

$$\int_{t=0}^{t=T} dX(t) = X(T) - X(0) = \int_{t=0}^{t=T} \mu\Big[t, X(t)\Big]dt + \int_{t=0}^{t=T} \sigma\Big[t, X(t)\Big]dz(t)$$

or

$$X(T) = X(0) + \int_{t=0}^{t=T} \mu\Big[t, X(t)\Big]dt + \int_{t=0}^{t=T} \sigma\Big[t, X(t)\Big]dz(t)$$

where the integral $\int_{t=0}^{t=T} \mu\Big[t, X(t)\Big]dt$ is called the pathwise ordinary integral and the integral $\int_{t=0}^{t=T} \sigma\Big[t, X(t)\Big]dz(t)$ is called the Itô stochastic integral.

3.1 Financial Models

Below a few popular financial models in terms of Lévy processes are discussed, particularly, the Black-Scholes model, Merton's jump-diffusion model, Kou model, and stochastic volatility models.

3.1.1 Black-Scholes Model

The Black-Scholes model is based on the assumption that the asset price follows a geometric Brownian motion SDE

$$dS = \mu S dt + \sigma S dz$$

Solving the SDE gives the dynamics of the asset price

$$S(T) = S(0)exp\Big[\Big(\mu - \frac{\sigma^2}{2}\Big)T + \sigma\epsilon\sqrt{T}\Big]$$

The Black-Scholes model assumes that the log-returns follow normal distribution with mean μ and variance σ^2, i.e., $L_1 \sim Normal(\mu, \sigma^2)$. The probability density function is then given by

$$fL_1(x) = \frac{1}{\sigma\sqrt{2\pi}}exp\Big[-\frac{(x-\mu)^2}{2\sigma^2}]$$

The characteristic function is

$$\varphi L_1(u) = exp\Big[i\mu u - \frac{\sigma^2 u^2}{2}]$$

The first and second moments are

$$E[L_1] = \mu, Var[L_1] = \sigma^2$$

The canonical decomposition of L is

$$L_t = \mu t + \sigma W_t$$

and the Lévy triplet is $(\mu, \sigma^2, 0)$.

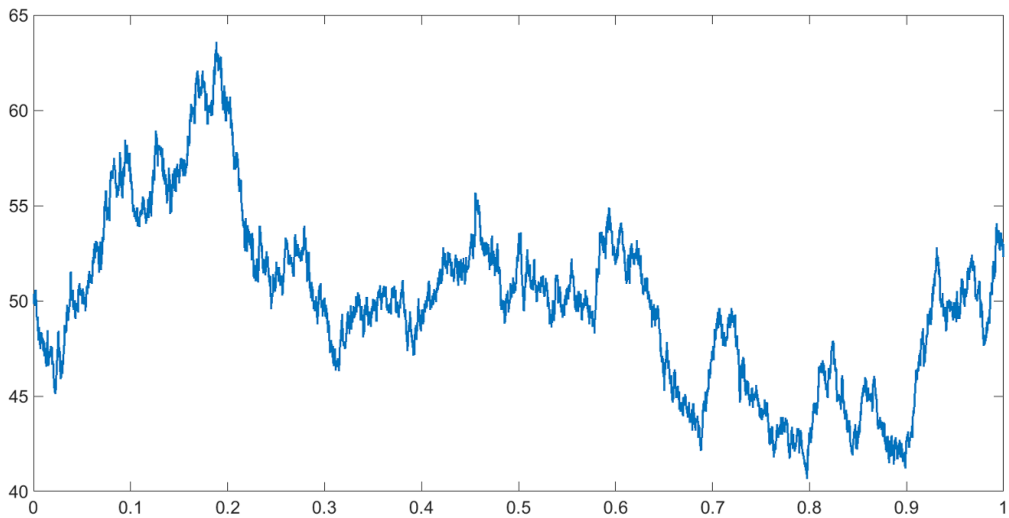

Figure 3.1: A Sample of Price Process under Black-Scholes Model with $S_0 = 50, \mu = 0.1, \sigma = 0.4$, and $T = 1$

3.1.2 Merton's Jump-Diffusion Model

The Black-Scholes model assumes that the underlying asset follows a diffusion process. However, Cox, Ross and Rubinstein in the famous binomial tree model for option pricing suggest that the underlying assets actually follow jumps rather than continuous change. Meanwhile, Robert C. Merton extended this approach, what is called Merton jump diffusion model since it combines jumps with diffusion terms.

The geometric Brownian motion Stochastic Differential Equation (SDE) is given by

$$dS = \mu S dt + \sigma S dz$$

or, equivalently

$$\frac{dS}{S} = \mu dt + \sigma dz$$

Robert Merton incorporated jumps governed by compound Poisson process into the diffusion model and introduced the mixed jump-diffusion model (Merton, 1976). In the Merton's model, the dynamics of the asset price is given by the SDE

$$\frac{dS}{S} = (\mu - \lambda k)dt + \sigma dz + dp$$

where μ is the expected return on the stock and σ is the volatility of the geometric Brownian motion. Here, λ is the average number of jumps per year and k is average jump size measured as a percentage of the asset price such that $k \equiv \epsilon(Y-1)$, where $(Y-1)$ is the random variable percentage change in the asset price if the Poisson event occurs and ϵ is the expectation operator over the random variable Y. Merton assumes that log stock price jump size follows normal distribution, $Normal(\mu, \delta^2)$, where δ is the volatility of jump size. Meanwhile, dz is the Brownian motion and dp is the Poisson process generating the jumps. The processes dz and dp are assumed to be independent. The probability of a jump in time Δt is $\lambda \Delta t$. The average growth in the asset price from the jumps is therefore λk.

If the Poisson event does not occur, i.e., $\lambda = 0$ and thus $dp \equiv 0$, the return dynamics is identical to that of the Black-Scholes model. If the Poisson event occurs, the return dynamics is given as

$$\frac{dS}{S} = (\mu - \lambda k)dt + \sigma dz + (Y-1)$$

where, with probability one, no more than one Poisson event occurs in an instant, and if the event does occur, then $(Y-1)$ is an impulse function producing a finite jump in $S(t)$ to $S(t)Y$. Solving the stochastic differential equation, the Merton's model, gives the dynamics of the asset price

$$S(t) = S(0)exp\Big[\Big(\mu - \frac{1}{2}\sigma^2 - \lambda k\Big)t + \sigma z(t)\Big]Y(n)$$

Here, $Y(n) = 1$ if $n = 0$ and $Y(n) = \sum_{j=1}^{n} Y_j$ for $n \geq 1$ where the Y_j are independently and identically distributed and n is Poisson distributed with parameter λt.

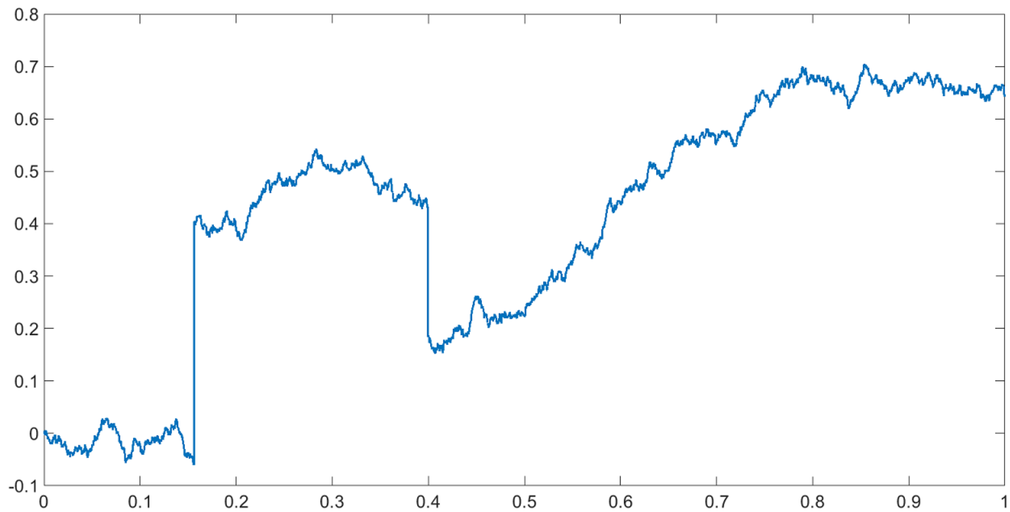

Figure 3.2: A Sample Price Process under Merton's Jump Diffusion Model with $\mu = 0.1, \sigma = 0.2, \lambda = 2, k = -0.1$, and $\delta = 0.1$

The canonical decomposition of the driving process is

$$L_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} J_k$$

where $J_k \sim Normal(\mu_k, \sigma_k^2), k = 1, \ldots$. The probability distribution of the jump size hence has density

$$f_j(x) = \frac{1}{\sigma_j \sqrt{2\pi}} exp\Big[- \frac{(x - \mu_j)^2}{2\sigma_j^2} \Big]$$

The characteristic function of L_1 is

$$\varphi_{L_1} = exp\Big[i\mu u - \frac{\sigma^2 u^2}{2} + \lambda(e^{(i\mu_j u - \sigma_j^2 u^2/2)} - 1) \Big]$$

and the Lévy triplet is $(\mu, \sigma^2, \lambda \times f_j)$. The density of L_1 is not known in closed form, while the first two moments are

$$E[L_1] = \mu + \lambda\mu_j, Var[L_1] = \sigma^2 + \lambda\mu_j^2 + \lambda\sigma_j^2$$

Figure 3.2 shows the simulated sample path of the Merton's jump diffusion model when drift rate $\mu = 0.1$, volatility $\sigma = 0.2$, average number of jumps per year $\lambda = 2$, average jump size $k = -0.1$, and volatility of jump size $\delta = 0.1$.

3.1.3 Kou Model

Steven Kou also proposed a jump-diffusion model, where the jump size is double-exponentially distributed (Kou, 2002). In this model, the dynamics of the asset price is given by the SDE

$$\frac{dS}{S} = \mu dt + \sigma dz + d\Big(\sum_{i=1}^{N(t)}(V_i - 1)\Big)$$

where z is a standard Brownian motion, $N(t)$ is a Poisson process with rate λ, and V_i is a sequence of independent identically distributed non-negative random variables such that $Y = log(V)$ has an asymmetric double exponential distribution with the density

$$f_Y(y) = p.n_1 e^{-n_1 y} 1_{y \geq 0} + q.n_2 e^{n_2 y} 1_{y<0}, n_1 > 1, n_2 > 0$$

where $p, q \geq 0$, $p+q = 1$ represent the probabilities of upward and downward jumps. In other words,

$$log(V) = Y = \begin{cases} \xi^+ & \text{with probability } p \\ -\xi^- & \text{with probability } q \end{cases}$$

where ξ^+ and ξ^- are exponential random variables with means $1/\eta_1$ and $1/\eta_2$, respectively. All sources of randomness, $N(t), z(t)$, and Ys are assumed to be independent. Solving this SDE gives the dynamics of the asset price

$$S(t) = S(0)exp\Big[\Big(\mu - \frac{1}{2}\sigma^2\Big)t + \sigma W(t)\Big]\sum_{i=1}^{N(t)} V_i$$

The canonical decomposition of the model is given by

$$L_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} J_k$$

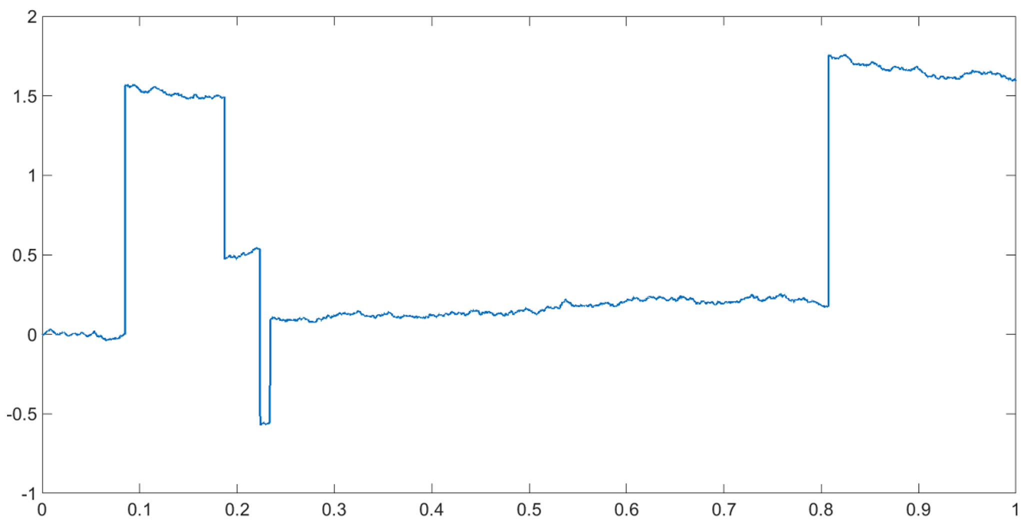

Figure 3.3: A Sample of Price Process under Kou model with $\mu = 0.1, \sigma = 0.2, \lambda = 2, p = 0.4, \eta_1 = 5$, and $\eta_2 = 3$

where $J_k \sim \text{DbExpo}(p, \theta_1, \theta_2), k = 1, \ldots$. The distribution of the jump size then has density

$$f_j(x) = p\theta_1 e^{-\theta_1 x} 1_{x<0} + (1-p)\theta_2 e^{\theta_2 x} 1_{x>0}$$

The characteristic function of L_1 is

$$\varphi_{L_1}(u) = exp\Big[i\mu u - \frac{\sigma^2 u^2}{2} + \lambda\Big(\frac{p\theta_1}{\theta_1 - iu} - \frac{(1-p)\theta_2}{\theta_2 + iu} - 1\Big)\Big]$$

and the Lévy triplet is $(\mu, \sigma^2, \lambda \times f_j)$. The first two moments are

$$E[L_1] = \mu + \frac{\lambda p}{\theta_1} - \frac{\lambda(1-p)}{\theta_2}, Var[L_1] = \sigma^2 + \frac{\lambda p}{\theta_1^2} + \frac{\lambda(1-p)}{\theta_2^2}$$

3.1.4 Stochastic Volatility Models

The Black Scholes model assumes that volatility remains constant over time. However, in practice, volatility varies through time and should be given as a stochastic variable (Wilmott, 2007). This results in financial models with two stochastic variables, the asset price and its volatility. Stochastic volatility models are currently popular for the pricing of contracts that are very sensitive to the behavior of the volatility. Among stochastic volatility models, one important model is given by the following two stochastic processes (Hull & White, 1987)

$$dS = rSdt + \sqrt{V}Sdz_S$$

$$dV = \mu V dt + \xi V dz_V$$

where S is the asset price, $V = \sigma^2$ is the instantaneous variance, r is the risk-free interest rate, μ is drift, and ξ is the volatility of the volatility. The two Brownian motions dz_S and dz_V of the asset price and variance, respectively, have correlation ρ.

Another famous stochastic volatility model is Heston model, and is given by the following coupled SDEs (Heston, 1993)

$$dS = rSdt + \sqrt{V}Sdz_S$$

$$dV = k(\theta - V)dt + \sigma\sqrt{V}dz_V$$

where S is the asset price, V is the instantaneous variance, r is the risk-free interest rate, k is the speed of mean-reversion, θ is the long-run variance, and σ is the volatility of the variance process. The two Brownian motions dz_S and dz_V of the asset price and variance, respectively, have correlation ρ. Here, k, θ and σ are constants over time.

An advancement to the Heston model is the Bates model which incorporates jumps into the diffusion model through a compound Poisson process (Bates, 1996). The Bates model is thus given by the following SDEs

$$\frac{dS}{S} = (r - \lambda k)dt + \sqrt{V}Sdz_S + dp$$

$$dV = k(\theta - V)dt + \sigma\sqrt{V}dz_V$$

where λ is the average number of jumps per year, k is the average jump size, and dp is the compound Poisson process. All other model parameters are the same as those of Heston's model.

3.1.5 GARCH Option Pricing Model

Duan introduced GARCH option pricing model where volatility is stochastic and changes over time (Duan, 1995). The risk-neutral asset price process of the GARCH model is given by the following coupled stochastic difference equations

$$\ln\frac{S_{t+1}}{S_t} = r - \frac{1}{2}\sigma_{t+1}^2 + \sigma_{t+1}\varepsilon_{t+1}$$

$$\sigma_{t+1}^2 = \beta_0 + \beta_1\sigma_t^2 + \beta_2\sigma_t^2(\varepsilon_t - \theta - \lambda)^2$$

This implies that

$$S_T = S_0\exp\Big[rT - \frac{1}{2}\sum_{s=1}^{T}\sigma_s^2 + \sum_{s=1}^{T}\sigma_s\varepsilon_s\Big]$$

where r is the risk-free interest rate, σ is the volatility, λ is the unit risk premium, ε is a standard normal random variable, and θ is a non-negative parameter that captures the negative correlation between asset return and volatility innovations. The $\beta_0, \beta_1, \beta_2$ are non-negative weights, where $\beta_0 + \beta_1 + \beta_2 = 1$.

3.1.6 Poisson-Diffusion Model

There are five option parameters: initial asset price S_0, strike price K, risk-free interest rate r, volatility of the asset price σ, and time to expiry T. All of the option parameters are almost known except for the future volatility of the asset price which is neither constant nor predictable, and not even directly observable. It makes volatility the single most important determinant of an option's value. The volatility time series show the volatility to be a highly unstable quantity. It is highly variable and unpredictable (Wilmott, 2007).

Markets move with investor expectations. One measure of future expectations is the CBOE's volatility index, VIX (Ackert, Kluger, & Qi, 2019). Similar to a bond's yield which represents the expected future return of the bond, the VIX reflects expected future market volatility as expressed through trade (Whaley, 2008). Figure 3.4 shows CBOE volatility index from December 2016 to February 2019 where volatility of the volatility index is too high during the year of 2018. The motivation here is then to propose a jump-diffusion model where volatility of asset's volatility is very high.

For high price volatility assets where large deviation from the mean price is expected, the following dynamics is proposed to model the asset price

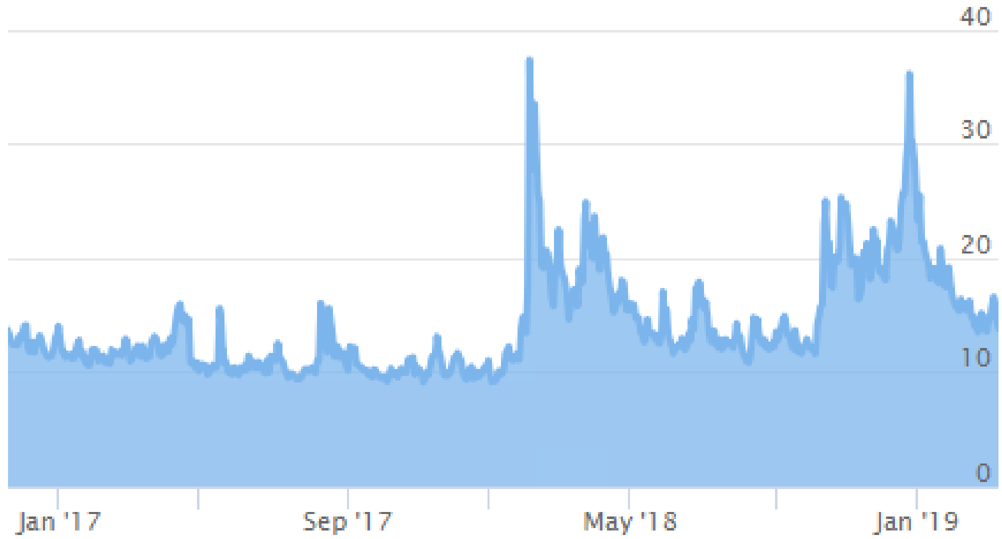

Figure 3.4: Chicago Board Options Exchange Volatility Index (Source: cboe.com)

$$\frac{dS}{S} = \beta_0\big[(r - \lambda k)dt + \sigma dz\big] + \beta_1 dp + \beta_2 dq$$

which is essentially the incorporation of Poisson process into the Merton's jump diffusion process with non-negative weights β_0, β_1, and β_2, where $\beta_0 + \beta_1 + \beta_2 = 1$. Here, r is the risk-free interest rate, λ is the average number of jumps per year, k is the average jump size, and σ is the volatility of the geometric Brownian motion. The processes dz, dp, and dq are Brownian motion, compound Poisson process, and Poisson process, respectively, and are assumed to be independent.

3.2 Weak Euler Scheme

The foundation of option pricing is based on the study of random walk of the underlying, the asset prices or interest rates. The dynamics of the underlying is modeled through an SDE which is a continuous time stochastic process, while simulations are done at discrete time steps. Hence, the first step in a simulation technique is to discretize an SDE. The first and the simplest numerical approximation method for SDEs is the Euler method (Maruyama, 1955).

Integrating the SDE $dX(t)$ over a very short interval of time from t to $t + dt$ gives

$$\int_t^{t+dt} dX(t) = X(d+dt) - X(t) = \int_t^{t+dt} \mu\Big[t, X(t)\Big] dt + \int_t^{t+dt} \sigma\Big[t, X(t)\Big] dz(t)$$

Hence, the Euler scheme, or Euler discretization, of the given SDE representing the change in X over the short time Δt is given by

$$X(t + \Delta t) = X(t) + \mu\Big[t, X(t)\Big]\Delta t + \sigma\Big[t, X(t)\Big]\sqrt{\Delta t} z$$

For financial derivatives and some other finance applications, weak order of convergence, defined by error in the expected value of payoff, is needed (Giles, 2012).

Specifically, for an arithmetic Brownian motion SDE, the drift coefficient and the diffusion coefficient are both constants. Hence, $\mu\Big[t, X(t)\Big] = \mu$ and $\sigma\Big[t, X(t)\Big] = \sigma$. The arithmetic Brownian motion SDE is thus given by

$$dX(t) = \mu dt + \sigma dz(t)$$

Integrating from time t to $t + dt$ gives

$$\int_t^{t+dt} dX(t) = X(d + dt) - X(t) = \int_t^{t+dt} \mu dt + \int_t^{t+dt} \sigma dz(t)$$

The Euler scheme representing the change in X over the short interval of time Δt is then given by

$$X(t + \Delta t) - X(t) = \mu\Delta t + \sigma\sqrt{\Delta t} z$$

or

$$X(t + \Delta t) = X(t) + \mu\Delta t + \sigma\sqrt{\Delta t} z$$

Accordingly, for a geometric Brownian motion SDE, the change in stochastic process, $dX(t)$, is with relation to the current value of $X(t)$. This proportional change $dX(t)/X(t)$ is modeled as an arithmetic Brownian motion

SDE. The SDE for a geometric Brownian motion, or an exponential Brownian motion, SDE is given by

$$\frac{dX(t)}{X(t)} = \mu dt + \sigma dz(t)$$

or equivalently

$$dX(t) = \mu X(t)dt + \sigma X(t)dz(t)$$

Integrating from time t to $t + dt$ gives

$$\int_t^{t+dt} dX(t) = X(d + dt) - X(t) = \int_t^{t+dt} \mu X(t)dt + \int_t^{t+dt} \sigma X(t)dz(t)$$

The Euler scheme representing the change in X over the short interval of time Δt is thus given by

$$X(t + \Delta t) = X(t) + \mu X(t)\Delta t + \sigma X(t)\sqrt{\Delta t}z$$

3.2.1 Black-Scholes Model

The Black-Scholes model assumes that the asset price S follows a geometric Brownian motion SDE where the asset's expected rate of return is the risk-free interest rate r. The geometric Brownian motion SDE in terms of S where $\mu = r$ is given by

$$dS(t) = rS(t)dt + \sigma S(t)dz(t)$$

and the Euler discretization is

$$S(t + \Delta t) = S(t) + rS(t)\Delta t + \sigma S(t)\sqrt{\Delta t}z$$

Practically, it is more precise to simulate $\ln S$ rather than S. Replacing $G = \ln S$, $x = S$, $a = rS$, and $b = \sigma S$ in

$$dG = (\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2)dt + \frac{\partial G}{\partial x}bdz$$

yields

$$d\ln S = (\frac{\partial \ln S}{\partial S}rS + \frac{\partial \ln S}{\partial t} + \frac{1}{2}\frac{\partial^2 \ln S}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial \ln S}{\partial S}\sigma S dz$$

Since

$$\frac{\partial}{\partial S}\ln S = \frac{1}{S}; \quad \frac{\partial^2}{\partial S^2}\ln S = -\frac{1}{S^2}; \quad \frac{\partial}{\partial t}\ln S = 0$$

then

$$d\ln S = (r - \frac{1}{2}\sigma^2)dt + \sigma dz$$

Equivalently

$$\ln S(t + \Delta t) - \ln S(t) = (r - \frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

Or, in exponential form

$$S(t + \Delta t) = S(t)\exp[(r - \frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon\sqrt{\Delta t}]$$

Here r is the risk-free interest rate, σ is the volatility, Δt is a small interval of time, and ε is a random sample from standard normal distribution. This equation is used to construct the path followed by the asset prices.

3.2.2 Hull & White Stochastic Volatility Model

The stochastic process for variance

$$dV = \mu V dt + \xi V dz_V$$

is a geometric Brownian motion SDE in V. The Euler discretization for $\ln V$ is given by

$$V(t + \Delta t) = V(t)\exp\left[\left(\mu - \frac{1}{2}\xi^2\right)\Delta t + \xi\varepsilon\sqrt{\Delta t}\right]$$

Similarly, the Euler discretization for the asset price SDE

$$dS = rSdt + \sqrt{V}Sdz_S$$

is given by

$$S(t + \Delta t) = S(t)\exp[(r - \frac{1}{2}V(t))\Delta t + \sqrt{V(t)}\varepsilon\sqrt{\Delta t}]$$

Or

$$S(t + \Delta t) = S(t)\exp[(r - \frac{1}{2}V(t))\Delta t + \varepsilon\sqrt{V\Delta t}]$$

Hull and White concluded that the Monte Carlo simulation method for option pricing can be used efficiently by assuming that the two Brownian motions are uncorrelated (i.e., $\rho = 0$). In other words, the asset price and volatility are uncorrelated, while allowing ξ and μ to depend on σ and t. This means that the instantaneous variance V follows a mean-reversion process

$$\mu = \alpha(\sigma^* - \sigma)$$

and it can be re-written as

$$dV = \alpha(\sigma^* - \sigma)V dt + \xi V dz_V$$

where α is the speed of mean-reversion, and σ^* is the long-run volatility. Here, α, σ^*, and ξ are constants over time.

3.2.3 Poisson-Diffusion Model

The Euler scheme for the Poisson-Diffusion Model introduced in Section 3.1.6 is given by

$$S(t + \Delta t) = S(t) + \beta_0\Big[\big(r - \frac{1}{2}\sigma^2 - \lambda k\big)\Delta t + \sigma\epsilon\sqrt{\Delta t}\Big] + \beta_1 P(i) + \beta_2 Q(i)$$

where $S(t + \Delta t)$ is a small change in the asset price from $S(t)$ during a small time interval Δt and ε is a random sample from the standard normal distribution. A large number of parameters makes it very flexible and allows better fit of the empirical distributions of the log-returns of financial data.

In the Poisson-Diffusion model, volatility remains constant over time, while the Poisson-Diffusion stochastic volatility model where the volatility varies through time is given by the following coupled SDEs

$$\frac{dS}{S} = rdt + \sqrt{V}dz_S$$

$$\frac{dV}{V} = \beta_0\big[(r - \lambda k)dt + \sigma dz_V\big] + \beta_1 dp + \beta_2 dq$$

The two Brownian motions dz_S and dz_V of the asset price and variance, respectively, are assumed to be independent. The Euler scheme for the Poisson-Diffusion stochastic volatility model is given as

$$S(t + \Delta t) = S(t)\text{exp}[(r - \frac{1}{2}V(t))\Delta t + \varepsilon\sqrt{V\Delta t}]$$

$$V(t+\Delta t) = V(t) + \beta_0\Big[\big(r - \frac{1}{2}\sigma^2 - \lambda k\big)\Delta t + \sigma\epsilon\sqrt{\Delta t}\Big] + \beta_1 P(i) + \beta_2 Q(i)$$

where ϵ is the random sample from the standard normal distribution.

Figure 3.5 shows simulations of the Poisson-Diffusion Model when drift rate $\mu = 0.15$, volatility $\sigma = 0.45$, the average number of jumps per year $\lambda = 1$, the average jump size $k = -0.1$, and volatility of jump size is $\delta = 0.1$. The assigned weights are $B_0 = 0.9$, $B_1 = 0.09999$, and $B_2 = 0.00001$.

Summary

In this chapter, the concept of SDEs in financial modeling is reviewed. Then, popular financial models, their Euler schemes and simulations are presented. The chapter also demonstrated how to incorporate jumps into the diffusion process. The systematic presentation of jump-diffusion models enabled to propose a new jump-diffusion model, referred to as the Poisson-Diffusion Model. Finally, the Poisson-Diffusion stochastic volatility model for assets with very high price volatility is proposed as a coupled SDEs.

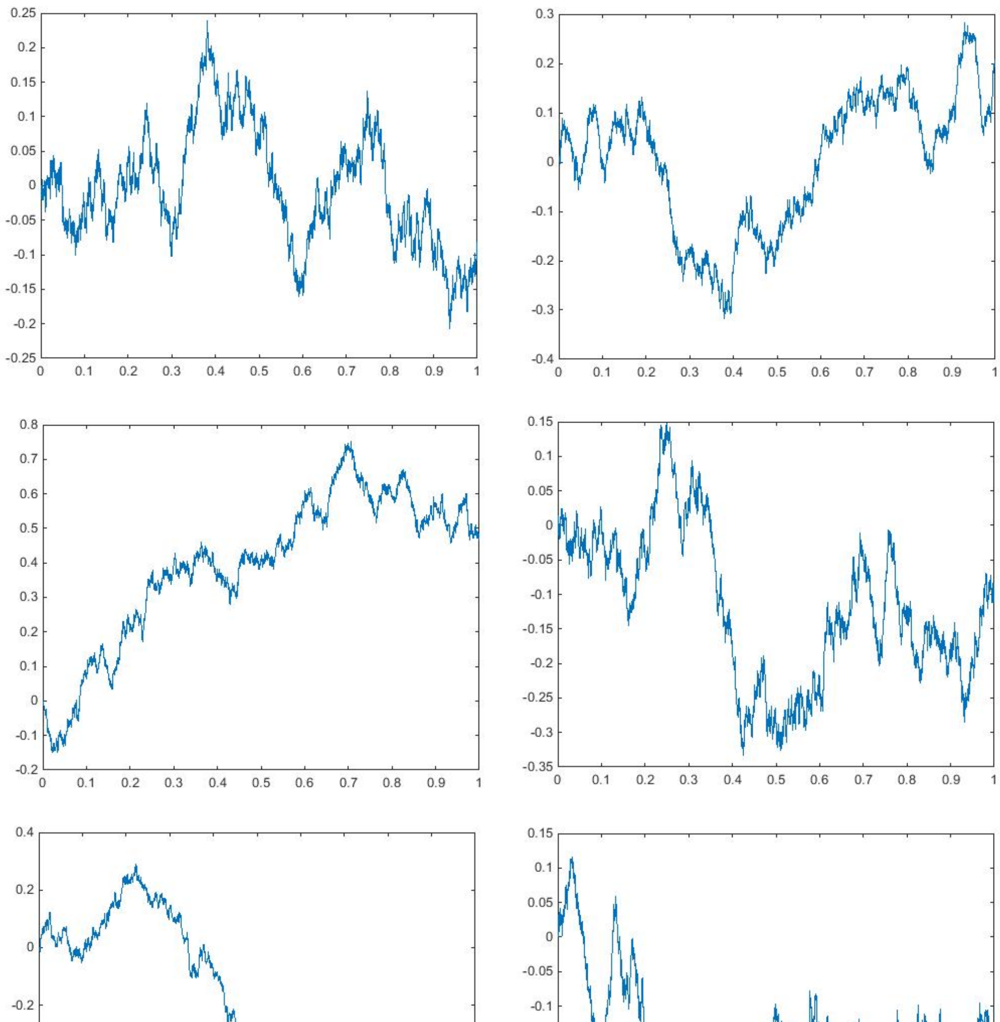

Figure 3.5: Simulations of Poisson-Diffusion model with $\mu = 0.15, \sigma = 0.45, \lambda = 1, k = -0.1,$, $\delta = 0.1$, $B_0 = 0.9$, $B_1 = 0.09999$, and $B_2 = 0.00001$

Chapter 4

Options

In general, there are two types of options, call and put. A call option is a contract which gives the holder of the option the right to buy the underlying asset by a certain date for an agreed price. Whereas, a put option gives the holder the right to sell the underlying asset by a certain date for an agreed price. Options that could only be exercised on the expiry date are called European style options, while options that could be exercised at any time up to the expiry date are called American style options.

This chapter begins with describing the Black-Scholes model for a standard option, and then presents the arithmetic and geometric Asian option formulas in the Black-Scholes world. The popularity of Asian options as a hedging instrument is then reviewed. Here, a new type of path-dependent options, referred to as the average-Asian options, is proposed. The calculations show that the average-Asian option reduces the underlying price volatility and is cheaper than the standard as well as Asian options in different practical scenarios.

4.1 Standard Options

The payoff from the standard European-style options, also called the plain vanilla products, depends on the price of the underlying at expiry, i.e., $t = T$. The payoff from a call is $\max(S_T - K, 0)$ and that from a put is $\max(K - S_T, 0)$, where S_T is the price of the underlying asset at expiry and K is called the strike or strike price.

The price of an option is a function of the underlying stock price S and time t, that is, $f = f(S, t)$. For S as given in the above equation, $f(S, t)$ satisfies the following equation (Itō, 1951)

$$df = (\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial f}{\partial S}\sigma S dz$$

A riskless portfolio of the option and stock can then be created to eliminate the Brownian motion from the above equation (Black & Scholes, 1973). This gives the Black-Scholes partial differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

In the risk-neutral valuation approach, the expected rate of return μ from an asset is the risk-free interest rate r. Hence, μ is replaced by r in the above equation.

The European call option gives the payoff $S - K$ at $t = T$ when $S > K$ and is worthless otherwise, so the terminal condition is

$$C(S,T) = \begin{cases} S - K & \text{if} \quad S > K \\ 0 & \text{if} \quad S \leq K \end{cases}$$

The option is worthless, that is, $C(0,t) = 0$ when $S = 0$. When the asset price increases without bound, that is, $S \to \infty$, the exercise becomes less and less important. Hence, the boundary conditions are

$$C(0,t) = 0$$

$$C(S,t) \sim S \quad \text{as} \quad S \to \infty$$

The solution to the Black-Scholes-Merton partial differential equation with these terminal and boundary conditions is the famous Black-Scholes-Merton option pricing formula for European-style call option

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

The functions $N(d_1)$ and $N(d_2)$ are the cumulative probability distribution function for a standardized normal distribution, where

$$d_1 = \frac{ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

By the put-call parity for European option

$$c - p = S_0 - e^{-rT} K$$

the formula for European put option is

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

Table 4.1 shows option prices of European style standard options with two different parameters.

Table 4.1: Option Prices with Different Parameters using Black-Scholes Option Pricing Formula

Parameters	Call	Put
$S_0 = 50, K = 50, r = 0.1, \sigma = 0.4, T = 1$	10.16	5.40
$S_0 = 42, K = 40, r = 0.1, \sigma = 0.2, T = 0.5$	4.76	0.81

4.2 Asian Options

For an Asian option, the payoff depends on the average price of the underlying asset during the life of the option. The payoff for an average price call option is

$$f = max(A(0,T) - K, 0)$$

and that for an average price put option is

$$f = max(K - A(0,T), 0)$$

where $A(0,T)$ is the average value of the underlying from $t = 0$ to $t = T$, inclusive, and K is the strike price. Two particular popularly used types are the geometric and arithmetic average options.

4.2.1 Geometric Asian Options

The geometric average is defined by

$$G(T) = \Big(\prod_{i=1}^{n} S(t_i)\Big)^{1/n}$$

and the continuously sampled geometric average is defined to be

$$G(T) = exp\Big(\frac{1}{T}\int_0^T S(\tau)d\tau\Big)$$

The Black-Scholes model relies on the assumption that the underlying price follows a lognormal distribution. The geometric average $G(T)$, whether discrete or continuous, of a lognormally distributed random variable is also lognormally distributed. The expectation and variance of $G(T)$ can then be derived to price the European style average call and put options (Kemna & Vorst, 1990). Specifically, for the continuous case,

$$logG(T) = n\Big(\frac{1}{2}\Big(r - \frac{1}{2}\sigma^2\Big)T + logS_0, \frac{1}{3}\sigma^2 T\Big)$$

Where $n(a, b)$ represents a normal distribution with mean a and variance b. Hence, the expectation and variance of $G(T)$ are given by

$$\mu_x = \frac{1}{2}\Big(r - \frac{1}{2}\sigma^2\Big)T + logS_0$$

and

$$\sigma_x^2 = \frac{1}{3}\sigma^2 T$$

Substituting them into the generalized version of Black's model (Black, 1976)

$$c = e^{-rT}\Big(e^{\mu_x + \frac{1}{2}\sigma_x^2} N\Big(\frac{\mu_x + \sigma_x^2 - logK}{\sigma_x}\Big) - KN\Big(\frac{\mu_x - logK}{\sigma_x}\Big)\Big)$$

yields the formula for geometric Asian call and put options, i.e.,

$$c = e^{-rT}\Big[F_0 N(d_1) - KN(d_2)\Big]$$

$$p = e^{-rT}\Big[KN(-d_2) - F_0 N(-d_1)\Big]$$

The functions $N(d_1)$ and $N(d_2)$ are the cumulative probability distribution functions for a standardized normal distribution, where

$$d_1 = \frac{ln(S_0 e^{aT}/K) + (\frac{1}{2}\sigma_A^2)T}{\sigma_A \sqrt{T}}$$

and

$$d_2 = d_1 - \sigma_A \sqrt{T}$$

where $a = \frac{1}{2}\Big(r - \frac{\sigma^2}{6}\Big)$, $\sigma_A = \frac{\sigma}{\sqrt{3}}$, and $F_0 = S_0 e^{aT}$.

4.2.2 Arithmetic Asian Options

The arithmetic average of the underlying, $A(0, T)$ is calculated as

$$A(0,T) = \frac{1}{N}\sum_{i=1}^{N} S(t_i)$$

The Black-Scholes model and Black's model rely on the assumption that the underlying price follows a lognormal distribution, while the arithmetic average of a lognormally distributed random variable is not lognormally distributed.

To value the arithmetic average Asian options by using Black's model, alternatively, it is assumed that $A(0, T)$ is lognormally distributed with respect to the first and second moments M_1 and M_2 (Turnbull & Wakeman, 2009), as shown in the following equation,

$$c = e^{-rT}\Big[F_0 N(d_1) - KN(d_2)\Big]$$

$$p = e^{-rT}\Big[KN(-d_2) - F_0 N(-d_1)\Big]$$

The functions $N(d_1)$ and $N(d_2)$ are the cumulative probability distribution functions for a standardized normal distribution, where

$$d_1 = \frac{ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}$$

$F_0 = M_1$, and

$$\sigma^2 = \frac{1}{T} ln\Big(\frac{M_2}{M_1^2}\Big)$$

where

$$M_1 = \frac{e^{rT} - 1}{rT} S_0$$

and

$$M_2 = \frac{2e^{[2r+\sigma^2]T} S_0^2}{(r+\sigma^2)(2r+\sigma^2)T^2} + \frac{2S_0^2}{rT^2}\Big(\frac{1}{2r+\sigma^2} - \frac{e^{rT}}{r+\sigma^2}\Big)$$

Using these formulas of arithmetic and geometric Asian options, the option prices with parameters $S_0 = 50, K = 50, r = 0.1, \sigma = 0.4$, and $T = 1$ are listed in Table 4.2.

Table 4.2: Arithmetic and Geometric Asian Option Prices with $S_0 = 50, K = 50, r = 0.1, \sigma = 0.4, T = 1$

Arithmetic		**Geometric**	
Call	**Put**	**Call**	**Put**
5.62	3.28	5.14	3.45

4.3 Average-Asian Options

4.3.1 Rationality

The averaging feature of the Asian options reduces the volatility inherent in the option, and makes them less exposed to price jumps. They are hence less expensive than the standard options. Since the creation of Asian options in 1987, they have been popular in financial markets (Chatterjee et al., 2018). The dependence of payoff structure on the average price of the underlying also makes them harder to be manipulated by large market participants. This is particularly important in case of thinly traded commodities (Linetsky, 2004). In general, Asian options are the most popular option contracts in the commodity market which is now a mainstream financial and investment class (Kyriakou, Pouliasis, & Papapostolou, 2016).

In corporate finance, the conflict of interests between shareholders, the principal who owns the company, and the top management, the agent who runs the company on behalf of the shareholders, is referred to as the principal-agent problem. In particular, shareholders want to maximize their wealth through increment in the share price, while the top management, or the executives, may look for corporate luxury, job security, or increment in their own wealth at the expense of the shareholders. Consequently, stock options are granted as an incentive paying to the executives to align the interests of the management and shareholders to mitigate the principal-agent problem in a company (Brealey et al., 2012). Still, despite their popularity, these options could not adequately align the interests of the two parties. Besides, there is also risk of stock price manipulations by the executives to boost their compensation packages (Hall & Murphy, 2003; Tian, 2017). In a survey of 169 Chief Financial Officers of U.S. public companies, It is reported that roughly 20% of the firms misrepresented their firm's economic performance and the main reason was the desire to influence the stock price (Dichev et al., 2013). It is then naturally recommended to utilize the averaging feature in designing executive stock options (ESOs) to better link the interests of the management and shareholders, as well as to preserve the value of options to both corporations and employees (Chhabra, 2008). This suggests that firms should consider granting Asian options instead of standard options as compensation packages (Tian, 2013). In addition, the payoff structure of an Asian option also resembles that of the variable annuity (Bernard, Cui, & Vanduffel, 2017), an insurance contract that is typically a long-term investment aimed at generating income for retirement.

By incorporating the average stock price into the payoffs of an ESO, which is now called an Asian executive option in the literature of executive compensations, it is shown that Asian options have advantages and are cheaper than the traditional options (Tian, 2013). Considering this advantage, power options in executive compensation were introduced where the option payoff is based on a power of the stock price at expiry. The payoff function of power option is given by (Bernard, Boyle, & Chen, 2016)

$$P_T = \psi\Big(S_T^{\varphi} - \frac{K}{\psi}\Big)^+$$

where $\varphi = \frac{1}{\sqrt{3}}$ and $\psi = S_0^{1-\varphi}\exp\Big\{\big(\frac{1}{2} - \varphi\big)\big(\mu - q - \frac{\sigma^2}{2}\big)T\Big\}$.

It is evidenced that the power options are even cheaper than the Asian options (Bernard, Boyle, & Chen, 2016). Both Asian and power executive options were priced in the Black-Scholes world. However, as the power option requires stock's expected return which is difficult to estimate and violates the risk-neutral valuation assumption of the Black-Scholes world, pricing these power options is more challenging than pricing Asian options (Bernard, Boyle, & Chen, 2016).

In addition, a payoff function as a weighted average of price at expiry and the average price from time $t = 0$ to $t = T$ is also considered (Jourdain & Sbai, 2007)

$$f\Big(\alpha S_T + \beta \int_0^T S_t dt\Big)$$

where α and β are non-negative constants such that $\alpha + \beta = 1$. Note that, for $\alpha = 0$, this is the payoff of an Asian option. According to the fundamental theorem of arbitrage-free pricing, the option price for this payoff function is

(Jourdain & Sbai, 2007)

$$C_0 = E\Big(e^{-rT} f\Big(\alpha S_T + \beta \int_0^T S_u du\Big)\Big)$$

Table 4.3: Option Prices Using Payoff Considered by Jourdain and Sbai with Percent Weights α and β

α	β	$r = 0.05, \sigma = 0.2$	$r = 0.1, \sigma = 0.3$	$r = 0.15, \sigma = 0.45$
0.0	1.0	5.75	9.03	13.09
0.1	0.9	6.14	9.67	14.08
0.2	0.8	6.56	10.38	15.09
0.3	0.7	6.99	11.10	16.14
0.4	0.6	7.43	11.83	17.28
0.5	0.5	7.91	12.65	18.36
0.6	0.4	8.39	13.43	19.55
0.7	0.3	8.90	14.22	20.75
0.8	0.2	9.39	15.04	21.95
0.9	0.1	9.96	15.86	23.19
1.0	0.0	10.46	16.73	24.44

The prices for the options considered by Jourdain and Sbai are given in Table 4.3. Here, the option parameters used are $S_0 = 100, K = 100, T = 1$ and three different combinations of r and σ. As expected and as shown by the table, the option is the cheapest when no weight is assigned to the S_T, that is, when $\alpha = 0$ and $\beta = 1$. The prices are keep increasing when more weight is assigned to S_T. Therefore, in order to make an option cheaper, this payoff is not useful. Another idea is to take the average of S_T and A_T, while in this case the payoff will always remain between standard and Asian option, that is, greater than the Asian and less than the standard option.

The variance of S_T is greater than that of A_T (Kemna & Vorst, 1990). It is a well admitted fact that the greater the volatility, or standard deviation, the greater the option value is. Then in the payoff function where the weighted average of S_T and A_T is taken, the variance of S_T needs to be reduced. This can be done by taking difference of the average of S_T and K with the K at expiry rather than the difference of S_T with K. Mathematically, $\frac{S_T+K}{2} - K$ instead of $S_T - K$. This further motivates to introduce the average-Asian option defined below. It is in line with the option considered by Jourdain and Sbai.

4.3.2 Definition and Properties

An average-Asian option is defined such as the payoff depends on the average of:

1. the average price of the underlying asset during the life of the option, and
2. the price of the underlying asset at expiry with the strike price.

The payoff from an average-Asian call is thus

$$f = \frac{1}{2}\max\Big(A_T - K + \frac{S_T + K}{2} - K, 0\Big) = \frac{1}{4}\max\Big(2A_T + S_T - 3K, 0\Big)$$

where S_T is the underlying price at expiry, A_T is the average price of the underlying asset from time $t = 0$ to $t = T$, and K is the strike price. Similarly, the payoff from an average-Asian put option is

$$f = \frac{1}{4}\max\Big(3K - (2A_T + S_T), 0\Big)$$

This option reduces the price volatility effectively. If the price is assumed to follow a geometric Brownian motion SDE, the expected mean and variance of S_T, A_T, and $AA_T = \frac{1}{4}(2A_T+S_T)$ are given in Table 4.4 for both the arithmetic and geometric averages. The SDE parameters considered are $S_0 = 100, r = 0.1, \sigma = 0.3, 0.4, 0.5, T = 1$, and $\Delta t = \frac{T}{m}$ where $m = 100$ and the simulations are performed for $n = 200,000$ times.

4.3.2.1 Expected Value

Consider the asset price $S(T)$ at $t = T$ and the average price $A(T)$ from time $t = 0$ to $t = T$. $A(T)$ and $S(T)$ can be written as

$$A(T) = \sum_{i=0}^{n} \frac{1}{n+1} S(T_i) \quad \text{and} \quad S(T) = \sum_{i=0}^{n} \frac{1}{n+1} S(T)$$

Table 4.4: Expected Means and Variances of S_T, A_T, and AA_T under a Geometric Brownian Motion Price Process

Parameter		S_T (Arithmetic)	A_T (Arithmetic)	AA_T (Arithmetic)	S_T (Geometric)	A_T (Geometric)	AA_T (Geometric)
Mean	$\sigma = 0.3$	110.37	105.12	80.15	110.48	103.75	79.49
	$\sigma = 0.4$	110.37	105.12	80.15	110.48	103.75	79.49
	$\sigma = 0.5$	110.37	105.12	80.15	110.48	103.75	79.49
Variance	$\sigma = 0.3$	1148.31	345.71	294.98	1150.18	330.88	287.55
	$\sigma = 0.4$	2111.82	626.50	537.71	2113.34	585.45	516.56
	$\sigma = 0.5$	3453.69	997.68	865.25	3458.86	915.22	822.99

Respectively, the value of the standard, Asian, and average-Asian call options are given by

$$C_{\text{standard}} = e^{-rT} E\Big[\max\Big(\sum_{i=0}^{n} \frac{1}{n+1} S(T) - K, 0\Big)\Big]$$

$$C_{\text{Asian}} = e^{-rT} E\Big[\max\Big(\sum_{i=0}^{n} \frac{1}{n+1} S(T_i) - K, 0\Big)\Big]$$

and

$$\begin{aligned} C_{\text{A.Asian}} &= e^{-rT} E\Big[\frac{1}{4}\max\Big(2\sum_{i=0}^{n} \frac{1}{n+1} S(T_i) + \sum_{i=0}^{n} \frac{1}{n+1} S(T) - 3K, 0\Big)\Big] \\ &= e^{-rT} E\Big[\frac{1}{2}\max\Big(\sum_{i=0}^{n} \frac{1}{n+1} S(T_i) - K + \sum_{i=0}^{n} \frac{\frac{S(T)+K}{2} - K}{n+1}, 0\Big)\Big] \end{aligned}$$

Proposition 4.1 *The expected value of the average-Asian call option is less than that of the standard call.*

Proof. Clearly,

$$E\Big[\max\Big(\sum_{i=0}^{n} \frac{\frac{S(T)}{n+1} + K}{2} - K, 0\Big)\Big] = \frac{1}{2} E\Big[\max\Big(\sum_{i=0}^{n} \frac{1}{n+1} S(T) - K, 0\Big)\Big]$$

$$< E\Big[\max\Big(\sum_{i=0}^{n}\frac{1}{n+1}S(T)-K,0\Big)\Big]$$

Also, by Kemna and Vorst (Kemna & Vorst, 1990),

$$E\Big[\max\Big(\sum_{i=0}^{n}\frac{1}{n+1}S(T_i)-K,0\Big)\Big] \leq E\Big[\max\Big(\sum_{i=0}^{n}\frac{1}{n+1}S(T)-K,0\Big)\Big]$$

Then, since either

$$E\Big[\frac{1}{2}\max\Big(\sum_{i=0}^{n}\frac{1}{n+1}S(T_i)-K+\sum_{i=0}^{n}\frac{\frac{S(T)+K}{2}-K}{n+1},0\Big)\Big] \leq E\Big[\max\Big(\sum_{i=0}^{n}\frac{1}{n+1}S(T_i)-K,0\Big)\Big]$$

or

$$E\Big[\frac{1}{2}\max\Big(\sum_{i=0}^{n}\frac{1}{n+1}S(T_i)-K+\sum_{i=0}^{n}\frac{\frac{S(T)+K}{2}-K}{n+1},0\Big)\Big] \leq E\Big[\max\Big(\sum_{i=0}^{n}\frac{\frac{S(T)+K}{2}-K}{n+1},0\Big)\Big]$$

it thus results in,

$$E\Big[\frac{1}{4}\max\Big(2\sum_{i=0}^{n}\frac{1}{n+1}S(T_i)+\sum_{i=0}^{n}\frac{1}{n+1}S(T)-3K,0\Big)\Big] < E\Big[\max\Big(\sum_{i=0}^{n}\frac{1}{n+1}S(T)-K,0\Big)\Big]$$

□

Remark 4.2 *At this point no definite relationship can be established between*

$$E\Big[max\Big(\sum_{i=0}^{n}\frac{1}{n+1}S(T_i)-K,0\Big)\Big] \quad and \quad E\Big[max\Big(\sum_{i=0}^{n}\frac{\frac{S(T)+K}{2}-K}{n+1},0\Big)\Big]$$

while by Kemna and Vorst (Kemna & Vorst, 1990), averaging reduces volatility by about 42%. In addition, considering Table 5.6, it could be argued that, when $K \leq S_0$

$$E\Big[max\Big(\sum_{i=0}^{n}\frac{\frac{S(T)+K}{2}-K}{n+1},0\Big)\Big] \leq E\Big[max\Big(\sum_{i=0}^{n}\frac{1}{n+1}S(T_i)-K,0\Big)\Big]$$

Then, a conjuncture for the case that $K \leq S_0$ could be

$$E\Big[\frac{1}{4}max\Big(2\sum_{i=0}^{n}\frac{1}{n+1}S(T_i)+\sum_{i=0}^{n}\frac{1}{n+1}S(T)-3K,0\Big)\Big] \leq E\Big[max\Big(\sum_{i=0}^{n}\frac{1}{n+1}S(T_i)-K,0\Big)\Big]$$

4.3.2.2 Payoff

To compare the payoffs from the Asian and Power options, Bernard, Boyle, & Chen (2016) used an example where they issued five year at-the-money and in-the money options on July 1, 2003 and also on July 1, 2008. For the sake of comparison, the same data (stock prices of Legg Mason Inc; Figure 4.1) are adopted to compare payoffs by different options. On July 1, 2003, the stock

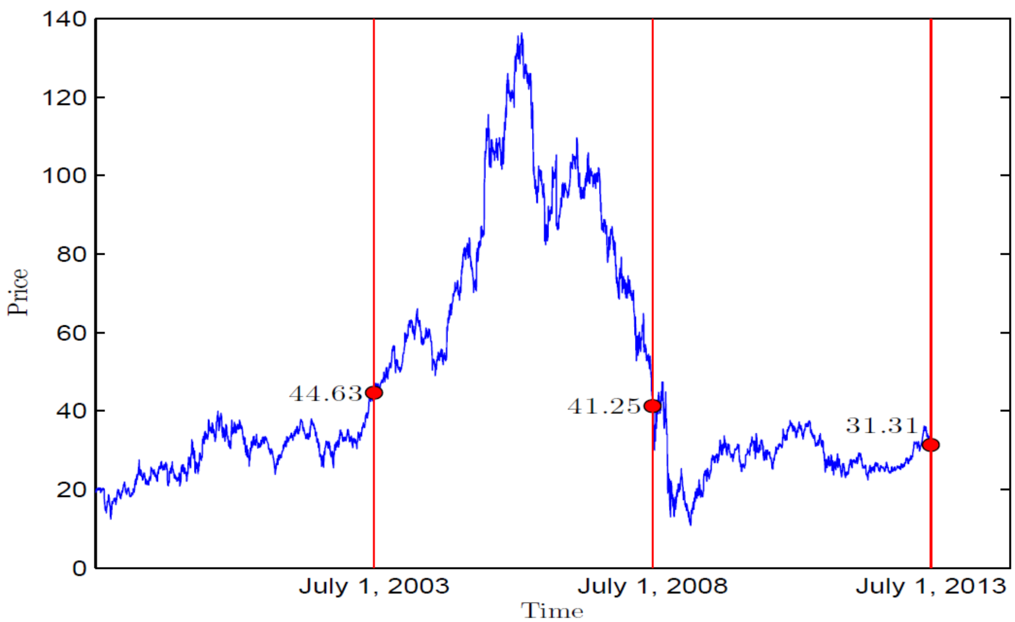

Figure 4.1: Legg Mason Closing Price from July 1, 1998 to July 1, 2013

price was 44.63 and it fell to 41.25 on July 1, 2008, while during this five year of time period, the price rose dramatically and has a high average price. The price further dropped to 31.31 on July 1, 2013. During this second five year of time period, the price decreased even further and had a low average price. The prices of two options, at-the-money (ATM) and in-the money (ITM), the strike K, price at expiry S_T, the average stock price A_T from July 1, 2003 to July 1, 2008 and July 1, 2008 to July 1, 2013 are given in Table 4.5.

In addition to the fact that the power option is complex and difficult to price (Bernard, Boyle, & Chen, 2016), as shown in Table 4.5, the payoff in power option is not consistent with the Asian option, which are popular hedging instruments and well understood by market participants. There are

Table 4.5: Payoff Comparison on Standard, Asian, Power, and Average-Asian Call Options

Issue	K	S_T	A_T	St.	Asian	Power	A.Asian
July 1, 2003 ATM	44.63	41.25	77.94	00	33.31	00	15.81
July 1, 2003 ITM	30	41.25	77.94	11.25	47.94	10.04	26.78
July 1, 2008 ATM	41.25	31.31	27.54	00	00	00	00
July 1, 2008 ITM	30	31.31	27.54	1.31	00	5.16	00

scenarios when the Asian option is in-the-money but the power option is out-of-money and vice versa. It would be a matter of concern for the holder of the power option. Whereas, the average-Asian option is consistent with the Asian option and is cheaper than the Asian option in the case when the average rose dramatically. Hence, the average-Asian option is a good choice as a hedging instrument where the asset price could rise or fall dramatically during the life of the option or where a cheaper option is sought.

Table 4.6 compares the payoffs of the standard, Asian, and average-Asian options in more detail where the initial asset price is assumed to be 100. It is clear that not only the payoff from the average-Asian option is less than that from the Asian option but also it is more stable in different practical situations. The standard deviation of the payoffs of the average-Asian option is always less than that of the Asian option. The dependence on both the price at expiry and the average price makes the average-Asian option less sensitive to sudden price jumps near the option expiry or during the life of the option.

4.3.3 Impact of Price Manipulation

Yisong S. Tian (2017) analyzed the sensitivity of different type of options to price manipulation in the executive compensation context, as well as calculated the potential gain from asset price manipulation both at the front-end, the time of option contract, and at the back-end, the time of option expiry.

In case of a call option, the front-end gaming involves a downward manipulation of the stock price in order to gain better terms, e.g., a lower exercise price for the call option. After the option contract, gains in the option payoff can only come from higher asset prices, which is called the back-end gaming. In case of a put option, the front-end gaming involves an upward manipu-

Table 4.6: Payoff Comparison on Standard, Asian, and Average-Asian Call Options

S_T	A_T	**St.**	**Asian**	**A.Asian**	**St.**	**Asian**	**A.Asian**
			$K = 100$			$K = 90$	
	140		40	25		50	32.5
	130		30	20		40	27.5
	120		20	15		30	22.5
120	110	20	10	10	30	20	17.5
	100		00	05		10	12.5
	90		00	00		00	7.50
	80		00	00		00	2.50
	70		00	00		00	00
SD			**15.81**	**9.80**		**19.59**	**11.76**
	130		30	17.5		40	25
	120		20	12.5		30	20
110	110	10	10	7.50	20	20	15
	100		00	2.50		10	10
	90		00	00		00	05
	80		00	00		00	00
SD			**12.65**	**7.19**		**16.33**	**9.35**
	120		20	10		30	17.5
	110		10	05		20	12.5
100	100	00	00	00	10	10	7.50
	90		00	00		00	2.50
	80		00	00		00	00
SD			**8.94**	**4.47**		**13.04**	**7.16**
	110		10	2.5		20	10
90	100	00	00	00	00	10	05
	90		00	00		00	00
SD			**5.77**	**1.44**		**10.00**	**5.00**

lation of the stock price, whereas the back-end gaming involves a downward manipulation of the stock price.

4.3.3.1 Front-End Gaming Sensitivity

If the asset price S_0 is manipulated by δ, then the asset price after manipulation is $S_1 = (1+\delta)S_0$ where δ is in percentage terms, i.e., $\delta = 0.05$ for 5% price manipulation, and is positive for upward manipulation and negative for downward manipulation. The front-end gaming sensitivity (FEGS) is thus given by

$$\text{FEGS} = \frac{C[S_0(1+\delta), K] - C[S_0, K]}{\delta C[S_0, K]}$$

Tian used an example to compare the sensitivities where a company announced that it will award its executives an option grant at an exercise price equal to the current stock price S_0 (Tian, 2017). It is assumed that the executives manipulate the asset price downward, in order to get lower strike price, to $S_1 = (1+\delta)S_0$ on or near the option grant date. The option price then reduces to $C_1 = C(S_1, K)$. Thus, the option gain translates into an FEGS measure of

$$\text{FEGS} = -\frac{C[S_0(1+\delta), K] - C[S_0, K]}{\delta C[S_0(1+\delta), K]}$$

Table 4.7 shows the FEGS measure for the standard, Asian, and Average-Asian options when the strike price is 90, 100, and 110, $S_0 = 100, r = 0.1, \sigma = 0.4$, and $T = 3$. The measure is given when δ is $1, 5, 10$, and 25 percent, respectively. Here, FEGS measure is negative because asset price is manipulated downward.

An FEGS measure of -1 means that a 1% downward manipulation in the asset price results in a 1% gain in the value of the option value. Interestingly, the Asian option is the most vulnerable to the front-end manipulation and the standard option is the least vulnerable. Whereas, the average-Asian option is less vulnerable to price manipulation threats than the Asian option.

4.3.3.2 Back-End Gaming Sensitivity

Once an option contract is written, gains in the option payoff can only come from manipulation in the asset price, normally, at the time of the option expiry, in case of a standard option, and during the life of the option, in case

Table 4.7: Front-End Gaming Sensitivity Measure with $S_0 = 100, r = 0.1, \sigma = 0.4$, and $T = 3$

δ	**-0.01**	**-0.05**	**-0.1**	**-0.25**
Standard				
$K = 90$	-1.9592	-2.1034	-2.3133	-3.2464
$K = 100$	-2.0615	-2.2184	-2.4475	-3.4765
$K = 110$	-2.1601	-2.3295	-2.5778	-3.7034
Asian				
$K = 90$	-3.0365	-3.3716	-3.8914	-6.6963
$K = 100$	-3.3716	-3.7701	-4.3953	-7.8832
$K = 110$	-3.7054	-4.1716	-4.9107	-9.1694
Average-Asian				
$K = 90$	-2.6736	-2.7732	-3.1283	-4.8741
$K = 100$	-2.4277	-2.9788	-3.3780	-5.4150
$K = 110$	-2.7401	-3.3145	-3.6624	-5.9327

of an Asian option. The option payoff sensitivity (OPS) (Tian, 2017) for different types of options is then

$$OPS_T = \frac{\max[S_T(1+\delta) - K, 0] - \max[S_T - K, 0]}{\delta C_0 \exp(rT)}$$

for a standard call,

$$OPS_T = \frac{\max[A_T(1+\delta)^{\Delta t/T} - K, 0] - \max[A_T - K, 0]}{\delta C_0 \exp(rT)}$$

for an Asian call, and

$$OPS_T = \frac{\max[2A_T(1+\delta)^{\Delta t/T} + S_T - 3K, 0] - \max[2A_T + S_T - 3K, 0]}{\delta C_0 \exp(rT)}$$

for an average-Asian call, where S_T is the asset price at expiry, A_T is the average price from time $t = 0$ to $t = T$, r is the interest rate, T is the life in years of the option, K is the agreed strike price, δ is the percentage change in the asset price due to manipulation, and C_0 is the option price at the time of

Table 4.8: Back-End Gaming Sensitivity Measure with $S_0 = 100, r = 0.1, \sigma = 0.4$, and $T = 3$

Δt	0.5			1		
δ	0.1	0.2	0.3	0.1	0.2	0.3
Standard						
$K = 90$	1.4559	1.4823	1.5032	1.4588	1.4833	1.5023
$K = 100$	1.5390	1.5737	1.5991	1.5370	1.5736	1.5996
$K = 110$	1.6154	1.6601	1.6990	1.6151	1.6606	1.6981
Asian						
$K = 90$	0.3320	0.3222	0.31301	0.6754	0.6634	0.6518
$K = 100$	0.3654	0.3560	0.3458	0.7436	0.7357	0.7251
$K = 110$	0.3960	0.3856	0.3762	0.8075	0.7999	0.7947
Average-Asian						
$K = 90$	0.2869	0.2779	0.2695	0.5825	0.5708	0.5592
$K = 100$	0.3097	0.3010	0.2918	0.6288	0.6194	0.6084
$K = 110$	0.3210	0.3207	0.3122	0.6709	0.6619	0.6540

option contract. Here, Δt is the manipulation period, that is, for how much time the price is manipulated, and $C_0\exp(rT)$ is the option price at $t = T$. As shown in Table 4.8, the standard option is the most vulnerable to the back-end gaming. If the option is granted at-the-money, i.e., $K = 100$, the BEGS measure is 1.5390, 1.5737, and 1.5991 when the percentage change in the asset price due to manipulation is 10, 20, and 30 percent, respectively, and the manipulation is maintained for 6 months. When the BEGS measure is 1.5390, it means that, 1% increase in the asset price can provide a 1.5390% gain in the expected payoff of the option. When δ is 10, 20, and 30 percent, respectively, and Δt is 6 months, the BEGS measure for the Asian option is 0.3654, 0.3560, and 0.3458, which is 76.25, 77.38, and 78.38 percent less sensitive than the corresponding standard option. Whereas, for these values of δ and Δt, the BEGS measure for the average-Asian option is 0.3097, 0.3010, and 0.2918, respectively. Hence, the average-Asian option is 79.87, 80.87, and 81.75 percent less sensitive than the corresponding standard option, and 15.24, 15.46, and 15.60 percent less sensitive than the corresponding Asian option. This indicates that the average-Asian option is the least sensitive to asset price manipulation at the back-end gaming. Hence, the average-Asian option is not only less sensitive than the Asian option against the front-end gaming but also against the back-end gaming.

Summary

In this chapter, the Black-Scholes model for the European style standard option is discussed, followed by the arithmetic and geometric Asian options. Then, the popularity of the Asian options for the purpose of risk management and executive compensation packages is discussed. The case where the price at expiry and the average price are given as the weighted average in the option payoff is also reviewed. Finally, with the intention to reduce the option price and the underlying price volatility, the average-Asian option is proposed. The numerical results demonstrate that the average-Asian option reduces the underlying price volatility and is less sensitive to price manipulations.

Chapter 5

Numerical Approximations for Option Pricing

The random behaviour of financial quantities is modeled through SDEs, which are continuous-time stochastic processes, while the numerical methods for solving these SDEs are based on a time discretization of the life T of the financial quantity into n time steps. Hence, the first step in the numerical methods is to discretize a continuous-time stochastic process into a discrete-time stochastic process.

This chapter explains the Euler method of discretizing the life T of a financial quantity into n time steps and updating its value at each step using a financial model. The option values are then calculated by using the Monte Carlo simulation method and sampling through a tree method. The option prices are then calculated using the stochastic volatility models including the newly proposed Poisson-Diffusion stochastic volatility model.

When the random behaviour of an asset is modeled through a geometric Brownian motion SDE for example, its discrete-time version is given by

$$S(t+\Delta t) = rS(t)\Delta t + \sigma S(t)\epsilon\sqrt{\Delta t}$$

Suppose a financial institute holds a 3 month option where the underlying is this asset, so $T = \frac{3}{12} = 0.25$. The initial price is 100, the risk-free interest rate is 5%, the volatility of the asset price is 30%, and the time T is divided into 100 equal intervals, i.e $\Delta t = \frac{T}{n} = \frac{0.25}{100} = 0.0025$. That is, $S_0 = 100, r = 0.05, \sigma = 0.3$, and $\Delta t = 0.0025$. The change ΔS in the current asset price, price at $t = 0$, $S_0 = 100$ during the time interval $\Delta t = 0.0025$ is then given by

$$S(t+\Delta t) = (0.05)(100)(0.0025) + (0.3)(100)(0.0835)\sqrt{0.0025}$$

which gives $S(t+\Delta t) = 0.1378$, and so $S_1 = 100 + 0.1378 = 100.1378$. Similarly, S_2, after a time of 0.0025+0.0025=0.005, can be obtained as

$$S(t+\Delta t) = (0.05)(100.1378)(0.0025) + (0.3)(100.1378)(0.52)\sqrt{0.0025}$$

which gives $S(t+\Delta t) = 0.7936$ and $S_2 = 100.1378+0.7936 = 100.9314$. Here, 0.1378 and 0.52 are normally distributed random numbers and are generated in Matlab through the >> *randn* command. This discrete-time method of modeling the asset price over time is called the Euler method.

The average-Asian option is contingent on both the price at expiry and the average price during the life of the option. In the absence of a closed-form solution, the sampling through tree and Monte Carlo simulation methods are probably the most suitable option pricing methodologies.

5.1 Sampling Through Tree Method

D. Mintz introduced the sampling through a tree method (Mintz, 1997). In this method, an m-step binomial tree is used to sample from the 2^m paths that are possible. Consider a binomial tree where the probability of an up movement is p and the probability of a down movement is $1-p$. The procedure is as follows:

1. At each node, a uniformly distributed random number between 0 and 1 is obtained. If the number is greater than or equal to $1-p$, it takes an up movement and down movement otherwise.

2. Once the path from the initial node ($t=0$) to the end of the tree ($t=T$) is complete, the price at expiry S_T is obtained and the average price A_T is calculated.

3. Calculate the payoff of the average-Asian option. The first trial is now complete.

4. A similar procedure from step 1 to step 3 is repeated to complete more trials.

5. The mean of the expected payoffs is discounted at the risk-free interest rate to obtain an approximate value of the average-Asian option.

Now, consider a three-month arithmetic average-Asian option with $S_0 = 100, r = 0.1, \sigma = 0.4, K = 100$, and $T = 0.25$. The life of the option is divided into seven intervals, that is, $\Delta t = \frac{T}{m} = \frac{0.25}{7} = 0.035714$. By the binomial model for option pricing (Cox, Ross, & Rubinstein, 1979),

$u = e^{\sigma\sqrt{\Delta t}} = e^{0.4\times\sqrt{0.035714}} = 1.0785$
$d = e^{-\sigma\sqrt{\Delta t}} = e^{-0.4\times\sqrt{0.035714}} = 0.9272$
$a = e^{r\sqrt{\Delta t}} = e^{0.1\sqrt{0.035714}} = 1.0036$

							169.749
						157.39	
					145.931		145.931
				135.306		135.306	
			125.455		125.455		125.455
		116.321		116.321		116.321	
	107.852		107.852		107.852		107.852
100		100		100		100	
	92.7194		92.7194		92.7194		92.7194
		85.9688		85.9688		85.9688	
			79.7097		79.7097		79.7097
				73.9063		73.9063	
					68.5255		68.5255
						63.5364	
							58.9105

Figure 5.1: Binomial Tree for Stock Price Movement with $m = 7$.

$p = \frac{a-d}{u-d} = \frac{1.0036-0.9272}{1.0785-0.9272} = 0.50475$
Then, the upward movement is
$100 \times 1.0785 = 107.852, \qquad 107.852 \times 1.0785 = 116.321, ...$
and the downward movement is
$100 \times 0.9272 = 92.7194, \qquad 92.7194 \times 0.9272 = 85.9688, ...$
The value of the Asian call is

$$c(\text{Asian}) = e^{-rT}5.70 = e^{-0.1\times 0.25}5.70 = 5.56$$

and the value of the average-Asian call option is

$$c(\text{A.Asian}) = e^{-rT}4.37 = e^{-0.1\times 0.25}4.37 = 4.26$$

The seven step binomial tree is given in Figure 5.1. The twenty trial paths are completed using the steps described above and are given in Table 5.2, where the last column is showing the payoff of the average-Asian option for each trial. The mean of these payoffs is 4.37, which is discounted at the risk-free interest rate to estimate the value of the average-Asian option.

In practice, more time steps, at least 30, on the tree and a sufficiently large number of simulation trials are needed to better approximate the option

Table 5.1: Pricing Asian and Average-Asian Options by Sampling through Tree Method with $S_0 = 100, r = 0.1, \sigma = 0.4, K = 100$, and $T = 0.25$

Trial	**Path**	S_T	**Average**	**Asian Payoff**	**A.Asian Payoff**
1	UUDUUDD	107.85	112.25	12.25	8.09
2	UUUDUUD	125.46	119.02	19.02	15.87
3	UDDUUUU	125.46	106.28	6.28	9.50
4	DUUUUUD	125.46	112.89	12.89	12.81
5	UDUDDDD	79.71	96.76	0	0
6	UUDUDDD	92.72	106.11	6.11	1.24
7	UUDDDUU	107.85	104.07	4.07	4.00
8	UDUUDDD	92.72	104.07	4.07	0.22
9	UDUDUDU	107.85	103.93	3.93	3.93
10	UUUUDDD	107.85	116.82	16.82	10.37
11	UDUDUDD	92.72	102.03	2.03	0
12	DUDDUUU	107.85	96.50	0	0.21
13	UDUUDUD	107.85	108.01	8.01	5.97
14	DUUUDUD	107.85	106.11	6.11	5.02
15	DDDUDUD	79.71	86.22	0	0
16	UDUUUDD	107.85	110.21	10.21	7.07
17	DUDUUUD	107.85	102.18	2.18	3.05
18	DDUDUUU	107.85	94.74	0	0
19	DDDUDUD	79.71	86.22	0	0
20	DDDUUUD	92.72	91.23	0	0
	Average			**5.70**	**4.37**

value. Table 5.3 shows the standard, Asian, and average-Asian arithmetic and geometric call option prices with $100,000$ simulations and for different time steps on tree with $S_0 = 100, r = 0.1, \sigma = 0.4, K = 100$, and $T = 1$.

Table 5.2: Prices of Standard, Asian, and Average-Asian Call Options by Sampling through Tree Method

Steps	Standard	Asian		Average-Asian	
		Arith	Geom	Arith	Geom
10	19.8961	11.0274	10.0623	10.3744	9.8690
20	20.2529	11.1948	10.1885	10.4758	9.9032
30	20.3325	11.1729	10.1646	10.4594	9.9257
40	20.2652	11.1199	10.1488	10.4014	9.8864
50	20.1302	11.0917	10.2380	10.3486	9.9365
60	20.3792	11.1841	10.2125	10.4486	9.8967
70	20.1256	11.0702	10.3403	10.3248	10.0314
80	20.1329	11.0147	10.1845	10.2976	9.8824
90	20.2719	11.1062	10.1909	10.3724	9.8711
100	20.2886	11.1208	10.2035	10.3837	9.9171

5.2 Monte Carlo Simulation

Updating the asset price from time $t = 0$ to $t = T$ using the Euler method as described above completes one sample path. In the Monte Carlo simulation method, a sufficiently large number of sample paths are generated to obtain the expected payoff of an option. Phelim Boyle in 1977 introduced the Monte Carlo simulation method for pricing options by simulating paths for the asset price in the risk-neutral world and calculating the payoff from the option (Boyle, 1977). The mean of the sample payoffs was discounted at the risk-free interest rate to approximate an option value. Consider the risk-neutral random walk for the asset price S

$$dS = rSdt + \sigma Sdz$$

whose Euler discretization is given by

$$S(t + \Delta t) = S(t) + rS(t)\Delta t + \sigma S(t)\epsilon\sqrt{\Delta t}$$

Phelim Boyle simulated $\ln S$ rather than S as given in Chapter 3, which gives the dynamics of the asset price as

$$S(t + \Delta t) = S(t)\exp[(r - \frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon\sqrt{\Delta t}]$$

From time $t = 0$ to $t = T$, it gives

$$S(T) = \exp[(r - \frac{1}{2}\sigma^2)T + \sigma\varepsilon\sqrt{T}]$$

The option value is therefore given as

$$\text{Option Value} = e^{-rT} E\Big[\text{Payoff}(S)\Big]$$

The procedure for valuing an option is thus outlined as follows:

1. Simulate the asset price S using the equation for $S(T)$ given above.
2. Calculate the option payoff.
3. Repeat steps 1 and 2 for a sufficiently large number of times to obtain the sample payoffs.
4. Calculate the arithmetic mean of these payoffs to approximate the expected option payoff.
5. Finally, discount the expected option payoff at the risk-free interest rate to estimate the option value.

The standard error of the approximations is calculated as

$$\text{Standard Error} = \frac{w}{\sqrt{M}}$$

where w is the standard deviation of the payoff results and M is the number of simulations or trials. Hence, a 95% confidence interval for the price f of an option is given as

$$\mu - \frac{1.96w}{M} < f < \mu + \frac{1.96w}{M}$$

The approximate value must be between this range at the 95% confidence interval.

The precision of the results by the Monte Carlo method critically depends on the number of simulations. Any technique that can reduce the variance is of great advantage. Antithetic variable technique is a simple and effective technique to reduce the variance. For example, a payoff f_1 is calculated from the stock price path constructed in the usual manner as above with

Table 5.3: Monte Carlo Simulation using Antithetic Variable Technique to Value Call Option with $S_0 = 50, r = 0.1, \sigma = 0.4, K = 52$, and $T = 1$

No. of Trials	**Call Value**	**Standard Error**	**Confidence Interval**
20,000	9.1972	0.0637	9.0723 - 9.3221
40,000	9.3113	0.0452	9.2227 - 9.3999
60,000	9.2194	0.0366	9.1477 - 9.2912
80,000	9.2479	0.0320	9.1851 - 9.3107
100,000	9.2711	0.0287	9.2148 - 9.3273
120,000	9.3067	0.0263	9.2552 - 9.3582
140,000	9.2414	0.0241	9.1942 - 9.2886
160,000	9.2492	0.0226	9.2049 - 9.2935
180,000	9.2424	0.0213	9.2006 - 9.2842
200,000	9.2614	0.0202	9.2217 - 9.3010

positive ε. According to the antithetic variable technique, another payoff f_2 is calculated from a second path constructed with negative ε, that is $-\varepsilon$. Then, the average of these two payoffs is taken, that is

$$f = \frac{f_1 + f_2}{2}$$

5.2.1 Black-Scholes Model

The binomial option pricing model assumes that the price movement follows a binomial distribution. For a large number of trials, the binomial distribution approaches the lognormal distribution assumed by the Black-Scholes option pricing model. The two resulting prices should then coincide (Cox, Ross, & Rubinstein, 1979). This is shown in the next pricing method when the average-Asian option is priced using the Monte Carlo simulation method.

Now assume that the asset price follows a geometric Brownian motion SDE (an assumption used by Black-Scholes model)

$$dS = rSdt + \sigma Sdz$$

And the solution to this SDE gives the dynamics of the asset price, that is

$$S(t + \Delta t) = S(t)\exp[(r - \frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon\sqrt{\Delta t}]$$

The average-Asian option price can then be calculated by using the Monte Carlo method. The process is described below.

1. Divide the time T of the option life by m, i.e., $\Delta t = \frac{T}{m}$.

2. Simulate the stock price using the solution to the geometric Brownian motion SDE and use the time step Δt instead of T.

3. Repeat step 2 m times by continuously updating S_0 by S_T for each time. Also, calculate the average price at each step.

4. After m times, the final price is S_T and the final average is A_T. Using this S_T and A_T, calculate the payoff of the average-Asian option.

5. This completes the first trial. Repeat steps 1 to 4 to perform n trials.

6. Calculate the average payoff from n trials and discount it to the risk-free interest rate. This is the average-Asian option price.

Table 5.4: Prices of Standard, Asian, and Average-Asian Call Options by Monte Carlo Simulation Method

Simulations	**Standard**	**Asian**		**Average-Asian**	
		Arith	**Geo**	**Arith**	**Geo**
20,000	20.5312	11.1156	10.1489	10.4341	9.8403
40,000	20.3281	11.1182	10.2921	10.3807	10.0160
60,000	20.2792	11.1153	10.2337	10.3690	9.9393
80,000	20.2698	11.1020	10.2313	10.3590	9.9382
100,000	20.3137	11.1185	10.3022	10.3800	9.9943
120,000	20.3345	11.1408	10.2092	10.3970	9.9126
140,000	20.2775	11.0800	10.2183	10.3514	9.9136

The option prices of the standard, Asian, and average-Asian call options are given in Table 5.4 when the number of time steps m is 100, i.e., $\Delta t = \frac{T}{m}$, and for different number of simulations, with $S_0 = 100, r = 0.1, \sigma = 0.4, K = 100$, and $T = 1$. Accordingly, Table 5.5 shows price comparison of these three types of different options by different pricing methodologies.

Table 5.5: Standard, Asian, and Average-Asian Option Prices Using Different Pricing Methods with $S_0 = 100, r = 0.1, \sigma = 0.4, K = 100$, and $T = 1$

Pricing Method	**Standard**	**Asian**		**Average-Asian**	
Black-Scholes	20.32	**Arith**	**Geo**	**Arith**	**Geo**
Sampling through Tree	20.27	11.12	10.25	10.38	9.94
Monte Carlo Method	20.33	11.11	10.26	10.38	9.95
Kemna and Vorst	...	...	10.27	...	...
Turnbull and Wakeman	...	11.23	...	...	...

Table 5.6 shows the prices of the standard, Asian, and average-Asian options with two different parameters and different strikes. From the table, it indicates that the resulting price from the average-Asian option is more stable in possible practical life as the strike price is very unlikely to be so large than the initial asset price. It is hence a good choice for reducing the volatility inherent in the option price.

The strike price, as shown in Table 5.6, remains an important factor in the determination of the option value comparison between the Asian and average-Asian options. The average-Asian call is fairly cheap compared with the Asian call when the option is in-the-money and slightly cheap when the option is at-the-money. With parameters $S_0 = 100, r = 0.15, \sigma = 0.45, T = 1$, when option is granted deep-in-the-money, i.e., $K = 90$, the standard option costs the firm \$29.5142 per share. Whereas, it costs the firm \$18.5279 per share if the Asian option is granted at this strike price. In the case of an average-Asian option, its cost is \$16.3836 per share, which is 55.511% and 88.426% of those of the standard and Asian options, respectively. Similarly, when the option is issued at-the-money, i.e., $K = 100$, the standard option costs the firm \$24.4169 per share. Whereas, it costs the firm \$13.0875, i.e., 53.6% of the standard option, and \$12.3747, i.e., 50.68% of the standard option and 94.55% of the Asian option, per share in the case of the Asian and average-Asian options, respectively. The average-Asian option is thus cheaper than the Asian option when the option is granted at-the-money also.

5.2.2 Stochastic Volatility Model

To price options in stochastic volatility model where volatility varies through time and is given as a stochastic variable, the Euler scheme for the Hull & White stochastic volatility model (Hull & White, 1987) is given by

$$S(t+\Delta t) = S(t)\text{exp}[(r - \frac{1}{2}V(t))\Delta t + \varepsilon\sqrt{V\Delta t}]$$

$$V(t+\Delta t) = V(t)\text{exp}\Big[\Big(\mu - \frac{1}{2}\xi^2\Big)\Delta t + \xi\varepsilon\sqrt{\Delta t}\Big]$$

The average-Asian option price can then be calculated by using the Monte Carlo simulation method. The process is described below.

1. Divide the time T of the option life by m, i.e., $\Delta t = \frac{T}{m}$.
2. Simulate the stock price using the above equation $S(t+\Delta t)$ where the variance V is the initial variance.
3. Update the variance using the above equation $V(t+\Delta t)$ and use this updated V to calculate $S(t+\Delta t)$ in the subsequent simulation.
4. Repeat steps 2 and 3 m times by continuously updating $S(t+\Delta t)$ and $V(t+\Delta t)$ each time. Also, calculate the average price at each step.
5. After m times, the final price is S_T and the final average is A_T. Using this S_T and A_T, calculate the payoff of the average-Asian option.
6. This is the completion of the first trial. Repeat steps 1 to 5 to perform n trials.
7. Calculate the average payoff from n trials and discount it to the risk-free interest rate. This is the average-Asian option price.

In Part A of Table 5.7, the variance V follows a mean-reversion process with speed $\alpha = 10$ and the long-run volatility $\sigma^* = 0.35$. Hence, μ in the variance process is replaced by $10(0.35 - \sqrt{V(t)})$. In Part B, the drift is allowed in the variance process as well. The other option parameters used in calculating the option prices are $S_0 = 100, K = 100, r = 0.15, T = 1$, and the volatility of the volatility is $\xi = 0.45$. It is constant throughout the life of the option. Here, the time T is divided into 100 steps, i.e., $m = 100$, and simulations are performed for $n = 200,000$ times. Note that when a drift in the stochastic volatility is allowed, the option prices, shown in Table 5.7, are higher compared to those, as shown in Table 5.6, from the model where the volatility is considered a constant.

5.2.3 Poisson-Diffusion Model

When jumps are incorporated into the price process $S(t + \Delta t)$, the standard option prices can be calculated in the usual manner using the Monte Carlo simulation method, while the Asian and average-Asian option prices do not converge to a single value. They are hence priced using the Poisson-Diffusion stochastic volatility model instead of the Poisson-Diffusion model, where jumps are incorporated into the variance process $V(t + \Delta t)$.

The Euler scheme for the Poisson-Diffusion stochastic volatility model is given by

$$S(t+\Delta t) = S(t)\mathrm{exp}[(r - \frac{1}{2}V(t))\Delta t + \varepsilon\sqrt{V\Delta t}]$$

$$V(t+\Delta t) = V(t) + \beta_0\Big[\big(r - \frac{1}{2}\xi^2 - \lambda k\big)\Delta t + \xi\epsilon\sqrt{\Delta t}\Big] + \beta_1 P(i) + \beta_2 Q(i)$$

The average-Asian option prices can be calculated by the Monte Carlo simulation method using the steps described above for the stochastic volatility model. Here, $P(i)$ and $Q(i)$ are compound Poisson and Poisson processes, respectively. All sources of randomness are assumed to be independent. In pricing options, the full truncation scheme (Lord, Koekkoek, & Dijk, 2010) is used for the stochastic volatility models to avoid negative variance. The asset price process thus takes the form

$$S(t+\Delta t) = S(t)\mathrm{exp}[(r - \frac{1}{2}V^+(t))\Delta t + \varepsilon\sqrt{V^+\Delta t}]$$

where $V^+ = \max(V, 0)$.

In Table 5.8, the prices of the standard, Asian, and average-Asian call options are calculated, in the case that the initial volatility of the underlying asset $\sigma = \sqrt{V}$ and the volatility of the volatility ξ first at 0.35 and then at 0.45, with a drift being allowed. The other option parameters are $S_0 = 100, r = 0.15, T = 1$, and the strike prices K are 90, 100, and 110. For the Poisson-Diffusion model, the average number of jumps per year is $\lambda = 1$, the average jump size is $k = -0.1$, and the volatility of the jump size is $\delta = 0.1$. Also, the assigned weights are $\beta_0 = 0.9, \beta_1 = 0.09999$, and $\beta_2 = 0.00001$.

Here, the time T is divided into 100 steps, i.e., $m = 100$, and simulations are performed for $n = 200,000$ times. Comparing the option prices in Table 5.8, where $\sigma = \xi = 0.45$, with those in Part B of Table 5.7, where drift is allowed, it can be noted that the prices are higher than the usual stochastic volatility model, with the incorporation of jumps in the variance process.

Summary

In this chapter, the standard, Asian, and average-Asian option prices are calculated using different financial models for both the cases when the volatility is constant and when it is stochastic during the life of the option. It is shown that the value of the average-Asian call option is consistently less than that of the standard call option and that of the Asian call when granted in-the-money or at-the money. Thus, the numerical results confirm the usefulness of the average-Asian options with the target being to reduce the underlying price volatility and the option price.

Table 5.6: Comparison of Standard, Asian, and Average-Asian Option Prices with Different Strikes

K	**European Option**	**Call**		**Put**	
	Average Type	**Arith**	**Geom**	**Arith**	**Geom**
	Part A. $S_0 = 100, r = 0.15, \sigma = 0.45, T = 1$**.**				
90	Standard	29.5142		6.9426	
90	Asian	18.5279	17.2313	3.1156	3.4871
90	Average-Asian	16.3836	15.7362	3.0346	3.2365
95	Standard	26.8550		8.6260	
95	Asian	15.6389	14.3920	4.5421	4.9772
95	Average-Asian	14.2697	13.6410	4.1641	4.3883
100	Standard	24.4169		10.5116	
100	Asian	13.0875	11.9187	6.3152	6.7892
100	Average-Asian	12.3747	11.7770	5.5122	5.736
105	Standard	22.1986		12.5417	
105	Asian	10.8866	9.7933	8.3809	8.9445
105	Average-Asian	10.7018	10.1411	7.0347	7.3064
110	Standard	20.0891		14.8014	
110	Asian	8.9695	7.9594	10.7885	11.4261
110	Average-Asian	9.1930	8.6825	8.7806	9.0774
	Part B. $S_0 = 50, r = 0.1, \sigma = 0.4, T = 1$**.**				
46	Standard	12.1803		3.8133	
46	Asian	7.7255	7.2453	1.7693	1.9463
46	Average-Asian	6.7821	6.5500	1.7123	1.8077
48	Standard	11.1109		4.5779	
48	Asian	6.5649	6.1158	2.4320	2.6300
48	Average-Asian	5.9336	5.7156	2.2339	2.3346
50	Standard	10.1553		5.4004	
50	Asian	5.5520	5.1188	3.2107	3.4382
50	Average-Asian	5.1856	4.9659	2.8262	2.9411
52	Standard	9.2438		6.3047	
52	Asian	4.6454	4.2441	4.1205	4.3741
52	Average-Asian	4.4994	4.2989	3.5022	3.6269
54	Standard	8.4276		7.2873	
54	Asian	3.8787	3.4960	5.1503	5.4265
54	Average-Asian	3.9043	3.7054	4.2550	4.3845

Table 5.7: Standard, Asian, and Average-Asian Option Prices Using Hull & White Stochastic Volatility Model

	Standard	**Asian**	**Average-Asian**
Part A.	$a = 10, \sigma^* = 0.35$		
$K = 90$	27.80	17.97	15.75
$K = 100$	22.44	12.39	11.59
$K = 110$	17.83	8.13	8.26
Part B.	Drift is allowed		
$K = 90$	29.83	18.60	16.48
$K = 100$	24.87	13.21	12.53
$K = 110$	20.56	9.10	9.36

Table 5.8: Standard, Asian, and Average-Asian Option Prices Using Poisson-Diffusion Stochastic Volatility Model

	K	**Standard**	**Asian**	**Average-Asian**
	90	29.50	17.93	16.04
$\sigma = 0.35$	100	24.15	12.09	11.79
	110	19.65	7.80	8.48
	90	31.38	19.02	17.03
$\sigma = 0.45$	100	26.36	13.59	13.06
	110	22.06	9.53	9.93

Chapter 6

Conclusion

In this thesis, a new path-dependent option, named the average-Asian option, is introduced, to further reduce the price of the option and to minimize the adverse effect of asset price jumps as well as the potential market manipulation threat. The average-Asian option is priced by using the Monte Carlo simulation method for both the cases when the volatility is constant and when it is stochastic during the life of the option. The resulting option prices are consistently more stable in the practical situations. The mathematical proof that the expected value of the average-Asian call is less than that of the standard call is given. The numerical results show that, on average, the average-Asian call option is about 49.3% cheaper than the standard call option when granted at-the-money. Through simulation, it is shown that, for $K \leq S_0$, the expected value of an average-Asian call is less than that of an Asian call. The numerical results show that, on average, the average-Asian call option is about 5.4% cheaper than the Asian call option when granted at-the-money, i.e., when $K = S_0$.

In executive compensation context, the average-Asian option is more cost effective than the Asian option both when the option is granted in-the-money and at-the-money. Besides, the average-Asian option is also less sensitive than the Asian option to managerial manipulation at both the front-end and the back-end gaming.

The price of crude oil is one of the leading indicators of the economy in forecasting the economic trends (Kyriakou, Pouliasis, & Papapostolou, 2016). An important example where the Asian option potentially becomes very expensive is the crude oil price from February 2018 to February 2019, given in Figure 6.1. The price in February 2018, as the initial price S_0, is greater than that in February 2019, as S_T. Whereas, the average price during this period is higher, which makes the Asian option an expensive choice. These are the particular situations where dependence on both the average price and the price at expiry, would have significant advantage over the average price alone.

As anything that reduces the up-front premium in an option contract makes it more popular (Wilmott, 2007), the average-Asian option would have significant practical importance in real life. The dependence of the

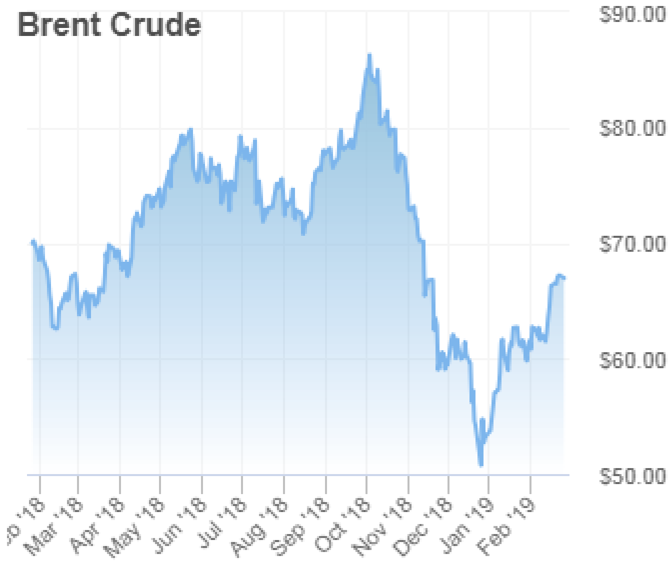

Figure 6.1: Brent Crude Oil Prices from February 2018 to February 2019 (Source: oilprice.com)

option payoff on both the average price of the underlying during the life of the option and the price at expiry makes the average-Asian option a powerful hedging instrument. Besides, as the Asian options are common in financial markets and well understood by market participants as well as academics, the average-Asian options would be easier to understand and trade, instead of being considered as abstract, technical, and complicated derivatives.

6.1 Research Implications

An Asian option is contingent on the average price of the underlying asset during the life of the option. The averaging feature reduces the price volatility which makes it less exposed to crashes or rallies in an asset price. Asian options are thus popular in the commodity market for the purpose of risk management.

It is evident from several numerical tests conducted in this research study that the Average-Asian option further reduces the underlying price volatility and also provides insurance against high average price but low price at the time of option exercise.

6.1.1 Theoretical Implications

Options are important derivatives and are widely used for the purpose of risk management. The average-Asian option is expected to be a valuable addition to the literature of derivatives. The dependence of the option payoff on both the average price of the underlying during the life of the option and the price at expiry makes the average-Asian option a powerful hedging instrument.

There are several useful financial models, with and without jumps, where only the asset price is considered as a stochastic variable, as well as where both the asset price and volatility are considered as stochastic variables. The proposed Poisson-Diffusion and Poisson-Diffusion stochastic volatility models are also expected to be useful models for highly volatile assets.

6.1.2 Practical Implications

In the energy market, for example the electricity or oil market, the contracts are written to supply continuous energy over the life of the option. The newly proposed path-dependent average-Asian options are particularly appropriate for such a market, as it is reasonable for the energy market to refer to the average price over the period of the contract while keeping the current market situation in account.

Another important applications of the average-Asian option is the renewable power production market where the weather condition, for example wind and cloud cover, has a strong effect on the supply side of the market. This could result in frequent price drops or rallies in the market. The design, i.e., dependence on both the average price and the price at expiry, and analysis, i.e., reduced volatility, of the average-Asian option make it a favorable financial derivative for both the producer and the consumer, in protecting against undesirable weather conditions and energy prices.

The average-Asian option is expected to be an effective risk management tool for businesses involved international trade and frequent currency exchange. It could be a simple but effective measure to avoid the extra risk imposed by the variability in the currency exchange rates. Also, it is expected to be a cost-effective tool to align the interest of management and shareholders and to mitigate the principal-agent problem in a firm.

6.2 Future Work

In section 4.4, it is argued that the expected value of an average-Asian call option is less than or equal to that of an Asian call option when $K \leq S_0$, using results shown in (Kemna & Vorst, 1990) and Table 5.6. An immediate step could then to ascertain a full mathematical relationship between the expected values of the Asian and average-Asian options.

Besides the models introduced in this thesis, the standard, Asian, and average-Asian option prices could also be calculated by using the GARCH option pricing model with Meixner innovations (Fengler & Melnikov, 2018). In addition, the average-Asian option could be further priced when the underlying is modeled as an SDE driven by a general meromorphic Lévy processes (Kuznetsov, Kyprianou, & Pardo, 2012). In the thesis, the basic Euler scheme is used to discretize the associated SDE, while the newly developed Euler-Poisson scheme (Ferreiro-Castilla, Kyprianou, & Scheichl, 2016) could be employed when the financial model is based on meromorphic Lévy processes. Furthermore, multilevel Monte Carlo (Giles, 2008) and empirical martingale (Duan & Simonato, 1998) are also promising simulation techniques for pricing the average-Asian option.

Options are used to hedge the underlying price risk. Similarly, Greeks, or Greek letters, are important in the hedging of an option position and play key roles in risk management (Hull, 2015; Wilmott, 2007). Greeks, i.e., delta, theta, gamma, vega, and rho, of the average-Asian option can hence also be calculated and analyzed. The Greeks of the Asian and average-Asian options could then be compared to measure the change in the option value with respect to the change in any individual option parameter.

In case of a geometric Asian option, when the underlying is assumed to follow a geometric Brownian motion SDE, the mean and variance are calculated and a closed-form formula is derived. An attempt could also be made to calculate the mean and variance of an average-Asian option in order to derive a closed-form formula for a geometric average-Asian option.

In section 3.7, a new stochastic process namely the Poisson-Diffusion model is proposed. The Poisson-Diffusion stochastic volatility model as a coupled SDEs is also given. However, most commodity prices tend to get pulled back to a long term mean and follow mean-reverting processes (Hull, 2015). the mean-reverting models are thus usually more popular to model commodity prices. Consequently, the mean-reversion form of the Poisson-

Diffusion and Poisson-Diffusion stochastic volatility models could also be designed and analyzed.

References

Ackert, L. F., Kluger, B. D., & Qi, L. (2019). Implied volatility and investor beliefs in experimental asset markets. *Journal of Financial Markets*, *43*, 121-136.

Bachelier, L. (2011). *Louis Bachelier's theory of speculation: The origins of modern finance*. Princeton, NJ: Princeton University Press.

Bates, D. S. (1996). Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. *The Review of Financial Studies*, *9*(1), 69-107.

Bernard, C., Boyle, P., & Chen, J. S. (2016). Power options in executive compensation. *Journal of Derivatives*, *23*(3), 9-20.

Bernard, C., Cui, Z., & Vanduffel, S. (2017). Impact of flexible periodic premiums on variable annuity guarantees. *North American Actuarial Journal*, *21*(1), 63-86.

Black, F. (1976). The pricing of commodity contracts. *Journal of Financial Economics*, *3*(1), 167-179.

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, *81*(3), 637-654.

Bodnar, G. M., Hayt, G. S., & Marston, R. C. (1998). Wharton survey of financial risk management by US non-financial firms. *Financial Management*, *27*(4), 70-91.

Boyle, P. P. (1977). Options: A monte carlo approach. *Journal of Financial Economics*, *4*(3), 323-338.

Boyle, P. P., & Emanuel, D. (1980). Options on the general mean. University of British Columbia, Canada, Working paper.

Brealey, R. A., Myers, S. C., Allen, F., & Mohanty, P. (2012). *Principles of corporate finance*. New York: Tata McGraw-Hill Education.

Brown, R. (1828). XXVII. A brief account of microscopical observations made in the months of June, July and August 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies. *The Philosophical Magazine*, *4*(21), 161-173.

CBOE (2019). CBOE global markets reports several trading volume records for full year 2018. Retrieved from http://ir.cboe.com/press-releases/2019/01-07-2019.

Chatterjee, R., Cui, Z., Fan, J., & Liu, M. (2018). An efficient and stable method for short maturity Asian options. *Journal of Futures Markets*, *38*(12), 1470-1486.

Chhabra, A. B. (2008). Executive stock options: Moral hazard or just compensation? *The Journal of Wealth Management*, *11*(1), 20-35.

Cont, R. (2001). Empirical properties of asset returns: Stylized facts and statistical issues. *Quantitative Finance*, *1*, 223-236.

Cont, R., & Tankov, P. (2004). *Financial modelling with jump processes.* London: Chapman & Hall/CRC Press LLC.

Cox, J. C., Ross, S. A., & Rubinstein, M. (1979). Option pricing: A simplified approach. *Journal of Financial Economics*, *7*(3), 229-263.

Dichev, I. D., Graham, J. R., Harvey, C. R., & Rajgopal, S. (2013). Earnings quality: Evidence from the field. *Journal of Accounting and Economics*, *56*(2, Supplement 1), 1-33.

Duan, J. C. (1995). The GARCH option pricing model. *Mathematical Finance*, *5*(1), 13-32.

Duan, J. C., & Simonato, J. G. (1998). Empirical martingale simulation for asset prices. *Management Science*, *44*(9), 1218-1233.

Einstein, A. (1911). Investigations on the theory of the Brownian movement. *Annal Physik*, *34*, 591-592.

Fanelli, V., Maddalena, L., & Musti, S. (2016). Asian options pricing in the day-ahead electricity market. *Sustainable Cities and Society*, *27*, 196-202.

Fengler, M. R., & Melnikov, A. (2018). GARCH option pricing models with Meixner innovations. *Review of Derivatives Research*, *21*(3), 277-305.

Ferreiro-Castilla, A., Kyprianou, A. E., & Scheichl, R. (2016). An Euler-Poisson scheme for Lévy driven stochastic differential equations. *Journal of Applied Probability*, *53*(1), 262-278.

Giles, M. B. (2008). Multilevel monte carlo path simulation. *Operations Research*, *56*(3), 607-617.

Giles, M. B. (2012). Euler scheme weak convergence. Advanced Monte Carlo methods. Department of Mathematics. University of Oxford.

Hall, B. J., & Murphy, K. J. (2003). The trouble with stock options. *Journal of Economic Perspectives*, *17*(3), 49-70.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, *6*(2), 327-343.

Hull, J. (2015). *Options, futures, and other derivatives.* Upper Saddle River, NJ: Pearson Education, Inc.

Hull, J., & White, A. (1987). The pricing of options on assets with stochastic volatilities. *The Journal of Finance*, *42*(2), 281-300.

Itô, K. (1951). *On stochastic differential equations.* (Vol. 4): American Mathematical Soc.

Jourdain, B., & Sbai, M. (2007). Exact retrospective Monte Carlo computation of arithmetic average Asian options. *Monte Carlo Methods and Applications*, *13*(2), 135-171.

Kemna, A. G. Z., & Vorst, A. C. F. (1990). A pricing method for options based on average asset values. *Journal of Banking & Finance*, *14*(1), 113-129.

Kou, S. G. (2002). A jump-diffusion model for option pricing. *Management Science*, *48*(8), 1086-1101.

Kuznetsov, A., Kyprianou, A. E., & Pardo, J. C. (2012). Meromorphic Lévy processes and their fluctuation identities. *The Annals of Applied Probability*, *22*(3), 1101-1135.

Kyprianou, A., Schoutens, W., & Wilmott, P. (2006). *Exotic option pricing and advanced Lévy models*. Chichester: John Wiley & Sons.

Kyriakou, I., Pouliasis, P. K., & Papapostolou, N. C. (2016). Jumps and stochastic volatility in crude oil prices and advances in average option pricing. *Quantitative Finance*, *16*(12), 1859-1873.

Linetsky, V. (2004). Spectral expansions for Asian (average price) options. *Operations Research*, *52*(6), 856-867.

Lord, R., Koekkoek, R., & Dijk, D. V. (2010). A comparison of biased simulation schemes for stochastic volatility models. *Quantitative Finance*, *10*(2), 177-194.

MacKenzie, D. (2008). *An engine, not a camera: How financial models shape markets*. Cambridge: MIT Press.

Maruyama, G. (1955). Continuous Markov processes and stochastic equations. *Rendiconti del Circolo Matematico di Palermo*, *4*(1), 48-90.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, *3*(1-2), 125-144.

Papapantoleon, A. (2008). An introduction to Lévy processes with applications in finance. Lecture notes. Department of Mathematics. Technical University Berlin.

Rubinstein, M., & Reiner, E. (1991). Unscrambling the binary code. *Risk Magazine*, *4*(9), 75-83.

Sauer, T. (2012). *Numerical solution of stochastic differential equations in finance Handbook of computational finance*. Berlin: Springer.

Schoutens, W. (2002). *The Meixner process: Theory and applications in finance*. Eurandom: Eindhoven.

STORE (2019). Stochastics of Renewable Energy Markets (STORE). Department of Mathematics. Retrieved from https://www.mn.uio.no/math/english/research/groups/store/index.html.

Tian, Y. S. (2013). Ironing out the kinks in executive compensation: Linking incentive pay to average stock prices. *Journal of Banking & Finance*, *37*(2), 415-432.

Tian, Y. S. (2017). Managerial gaming of stock and option grants. *Financial Markets, Institutions & Instruments*, *26*(3), 127-152.

Turnbull, S. M., & Wakeman, L. M. (2009). A quick algorithm for pricing European average options. *Journal of Financial and Quantitative Analysis*, *26*(3), 377-389.

Whaley, R. E. (2008). Understanding vix. *Journal of Portfolio Management*, *35*(3), 98-105.

Wiener, N. (1921). The average of an analytic functional and the Brownian movement. *Proceedings of the National Academy of Sciences*, *7*(10), 294-298.

Wilmott, P. (2007). *Paul Wilmott introduces quantitative finance.* Chichester: John Wiley & Sons.

Appendices

Matlab Codes

Sampling through Tree Method for Option pricing

```
S=100; K=100; r=0.1; V=0.4; T=1; N=100; M=200000;
sum=0; BSPayoffSum=0; AsianPayoffSum=0; AAsianPayoffSum=0;
dt=T/N;
a = exp(r*dt);
U = exp(V*sqrt(dt));
D = exp(-V*sqrt(dt));
P = (a-D)/(U-D);
Q = 1 - P;
for J=1:M
    S = 100;
sum = 100;
for k=1:N
if rand >=Q
    S = S*U;
else
    S = S*D;
end
sum = sum+S; % For geometric: sum = sum*S;
end
A = sum/(N+1); % For geometric: A = (sum)^(1/(N+1));
BSPayoff = max(S-K,0);
AsianPayoff = max(A-K,0);
AAsianPayoff = 0.25*max(2*A+S-(3*K),0);
BSPayoffSum = BSPayoffSum + BSPayoff;
AsianPayoffSum = AsianPayoffSum + AsianPayoff;
AAsianPayoffSum = AAsianPayoffSum + AAsianPayoff;
end
BSPayoffAverage = BSPayoffSum/M;
BSCall = exp(-r*T)*BSPayoffAverage
AsianPayoffAverage = AsianPayoffSum/M;
AsianCall = exp(-r*T)*AsianPayoffAverage
AAsianPayoffAverage = AAsianPayoffSum/M;
```

```
AAsianCall = exp(-r*T)*AAsianPayoffAverage
```

Monte Carlo Simulation for Option pricing

```
S=100; K=100; r=0.1; V=0.4; T=1; m=500; n=200000; dt=T/m;
BSPayoffSum=0; AsianPayoffSum=0; AAsianPayoffSum=0;
for I=1:n
    S = 100;
STSum = 100;
    AT = 100;
for J=1:m
    ST = S*exp(((r-V^2/2)*dt)+(V*randn*sqrt(dt)));
STSum = STSum + ST; % For geometric: STSum = STSum*ST;
    AT = STSum/(J+1); % For geometric: AT = (STSum)^(1/(J+1));
    S = ST;
end
BSPayoff = max(ST-K,0);
AsianPayoff = max(AT-K,0);
AAsianPayoff = 0.25*max(2*AT+ST-(3*K),0);
BSPayoffSum = BSPayoffSum + BSPayoff;
AsianPayoffSum = AsianPayoffSum + AsianPayoff;
AAsianPayoffSum = AAsianPayoffSum + AAsianPayoff;
end
BSCall = exp(-r*T)*(BSPayoffSum/n)
AsianCall = exp(-r*T)*(AsianPayoffSum/n)
AAsianCall = exp(-r*T)*(AAsianPayoffSum/n)
```

Hull & White SV Model with Mean-Reversion

```
function Asian = SVModel(so,k,r,vo,t,m,n)
dt = t/m; BSPayoffSum=0; AsianPayoffSum=0; AAsianPayoffSum=0;
for i=1:n
s = so;
v = vo^2;
dv = vo^2;
stSum = so;
at = so;
for j=1:m
```

```
st = s*exp(((r-1/2*dv)*dt)+(randn*sqrt(dv*dt)));
dv = v*exp((10*(0.35 - sqrt(v)) - 0.5*vo^2)*dt+vo*randn*sqrt(dt));
stSum = stSum + st;
at = stSum/(j+1);
s = st;
v = dv;
end
BSPayoff = max(st-k,0);
BSPayoffSum = BSPayoffSum + BSPayoff;
AsianPayoff = max(at-k,0);
AsianPayoffSum = AsianPayoffSum + AsianPayoff;
AAsianPayoff = 0.25*max(2*at+st-3*k,0);
AAsianPayoffSum = AAsianPayoffSum + AAsianPayoff;
end
BSCall = exp(-r*t)*(BSPayoffSum/n)
AsianCall = exp(-r*t)*(AsianPayoffSum/n)
AAsianCall = exp(-r*t)*(AAsianPayoffSum/n)
```

Poisson Random Generator

```
function y = Poisson(lambda)
X = 0;
Sum = 0;
flag = 0;
while flag == 0
E = -log(rand);
Sum = Sum + E;
if Sum < lambda
X = X + 1;
else
flag = 1;
end
end
y = X;
```

Poisson-Diffusion Model

```
function PoissonDiffusionModel(mu,sigma,lambda,k,delta)
```

```
T = 1;
N = 5000;
dt = T/N; t = (0:dt:T);
I = zeros(N,1);
X = zeros(N+1,1); Q = zeros(N+1,1); X(1) = 0;
P = zeros(N+1,1);
for i = 1:N
I(i) = Poisson(dt*lambda);
Q(i+1) = Q(i) + I(i);
if I(i) == 0;
P(i) = 0;
else P(i) = k*I(i) + sqrt(delta)*sqrt(I(i))*randn;
end
X(i+1) = X(i) + 0.9*((mu-0.5*sigma^2-lambda*k)*dt +
sigma*sqrt(dt)*randn) + 0.09999*P(i) + 0.00001*Q(i);
end
plot(t,X)
```

Poisson-Diffusion Stochastic Volatility Model

```
function PoissonDiffusionSV(so,k,r,v,t,lambda,k,delta)
m = 100; n = 100000; dt = t/m;
BSPayoffSum = 0; AsianPayoffSum = 0; AAsianPayoffSum = 0;
I = zeros(m,1);
X = zeros(m+1,1); Q = zeros(m+1,1);
P = zeros(m+1,1);
for j=1:n
    X(1) = v^2;
    s = so;
    stSum = so;
    at = so;
for i = 1:m
I(i) = Poisson(dt*lambda);
Q(i+1) = Q(i) + I(i);
if I(i) == 0;
P(i) = 0;
else P(i) = k*I(i) + sqrt(delta)*sqrt(I(i))*randn;
end
```

```
st = s*exp(((r-1/2*max(0,X(i+1)))*dt)+(randn*sqrt(max(0,X(i+1))*dt))
X(i+1) = X(i) + 0.9*((r-0.5*v^2-lambda*k)*dt + v*sqrt(dt)*randn) +
0.09999*P(i) + 0.00001*Q(i);
stSum = stSum + st;
at = stSum/(m+1);
s = st;
end
BSPayoff = max(st-k,0);
BSPayoffSum = BSPayoffSum + BSPayoff;
AsianPayoff = max(at-k,0);
AsianPayoffSum = AsianPayoffSum + AsianPayoff;
AAsianPayoff = 0.25*max(2*at+st-3*k,0);
AAsianPayoffSum = AAsianPayoffSum + AAsianPayoff;
end
BSCall = exp(-r*t)*(BSPayoffSum/n)
AsianCall = exp(-r*t)*(AsianPayoffSum/n)
AAsianCall = exp(-r*t)*(AAsianPayoffSum/n)
```